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AN EFFICIENT DERIVATIVE FREE ITERATIVE METHOD FOR SOLVING SYSTEMS OF NONLINEAR EQUATIONS

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We present a derivative free method of fourth order convergence for solving systems of nonlinear equations. The method consists of two steps of which first step is the well-known Traub's method. First-order divided difference operator for functions of several variables and direct computation by Taylor's expansion are used to prove the local convergence order. Computational efficiency of new method in its general form is discussed and is compared with existing methods of similar nature. It is proved that for large systems the new method is more efficient. Some numerical tests are performed to compare proposed method with existing methods and to confirm the theoretical results.

1. INTRODUCTION

The problem of finding solutions of the system of nonlinear equations $F(x) = 0$, where $F : D \to D$, D is an open convex domain in R^m, by iterative methods is an important and interesting task in numerical analysis and applied scientific branches. One of the basic procedures for solving nonlinear equations, is the quadratically convergent Newton's method (see [**8, 10**]),

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \quad k = 0,\, 1,\, 2, \ldots,$$

where $[F'(x)]^{-1}$ is the inverse of first Fréchet derivative $F'(x)$ of the function $F(x)$.

In many practical situations it is preferable to avoid the calculation of derivative $F'(x)$ of the function $F(x)$. In such situations a method that uses only the

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computed values of $F(x)$ is more appropriate. For example, a basic derivative free iterative method is the TRAUB's method [**20**], which also converges quadratically and follows the scheme

$$x^{(k+1)} = \mathrm{M}_{2,1}(x^{(k)}) = x^{(k)} - [w^{(k)}, x^{(k)}\,; F]^{-1} F(x^{(k)}), \tag{1}$$

where $[w^{(k)}, x^{(k)}\,; F]^{-1}$ is the inverse of the first order divided difference $[w^{(k)}, x^{(k)}\,; F]$ of F and $w^{(k)} = x^{(k)} + \beta F(x^{(k)})$, β is an arbitrary non-zero constant. Here, $\mathrm{M}_{p,i}$ is used for denoting i^{th} method of convergence order p. For $\beta = 1$ the scheme (1) reduces to the well-known STEFFENSEN's method [**19**].

To solve a scalar equation $f(x) = 0$, many higher order efficient methods have been proposed in literature which do not involve derivative of the function $f(x)$, see [**3, 9, 12, 13, 14, 15, 17, 18, 21, 23**] and references therein. However, for systems of nonlinear equations higher order derivative free methods are very rare in literature. Recently, based on STEFFENSEN's scheme, i.e. when $\beta = 1$ in (1), a family of seventh order methods has been proposed in [**21**]. Some important methods of this family are

$$\begin{aligned} y^{(k)} &= \mathrm{M}_{2,1}(x^{(k)}), \\ z^{(k)} &= \mathrm{M}_{4,1}(x^{(k)}, y^{(k)}) \\ &= y^{(k)} - ([y^{(k)}, x^{(k)}\,; F] + [y^{(k)}, w^{(k)}\,; F] - [w^{(k)}, x^{(k)}\,; F])^{-1} F(y^{(k)}), \\ x^{(k+1)} &= \mathrm{M}_{7,1}(x^{(k)}, y^{(k)}, z^{(k)}) \\ &= z^{(k)} - ([z^{(k)}, x^{(k)}\,; F] + [z^{(k)}, y^{(k)}\,; F] - [y^{(k)}, x^{(k)}\,; F])^{-1} F(z^{(k)}) \end{aligned} \tag{2}$$

and

$$\begin{aligned} y^{(k)} &= \mathrm{M}_{2,1}(x^{(k)}), \\ z^{(k)} &= \mathrm{M}_{4,2}(x^{(k)}, y^{(k)}) = y^{(k)} - [y^{(k)}, x^{(k)}\,; F]^{-1} \\ &\times ([y^{(k)}, x^{(k)}\,; F] - [y^{(k)}, w^{(k)}\,; F] + [w^{(k)}, x^{(k)}\,; F])[y^{(k)}, x^{(k)}\,; F]^{-1} F(y^{(k)}), \\ x^{(k+1)} &= \mathrm{M}_{7,2}(x^{(k)}, y^{(k)}, z^{(k)}) \\ &= z^{(k)} - ([z^{(k)}, x^{(k)}\,; F] + [z^{(k)}, y^{(k)}\,; F] - [y^{(k)}, x^{(k)}\,; F])^{-1} F(z^{(k)}). \end{aligned} \tag{3}$$

These algorithms are notable not only for their simple structure but also their efficiency. The fourth order scheme $\mathrm{M}_{4,1}(x^{(k)}, y^{(k)})$ is the generalization of the method proposed by REN et al. [**17**] whereas $\mathrm{M}_{4,2}(x^{(k)}, y^{(k)})$ is the generalization of the method by LIU et al. [**9**].

In this paper, our aim is to develop a derivative free method with higher convergence order and minimum computational cost. In order to achieve this goal, we here propose a method with fourth order of convergence by considering the structure of the scheme in such a way that it utilizes as minimum number of function evaluations as possible so that it may have low computational cost. Thus, we show that the present method is more efficient than existing derivative free methods, particularly for larger systems.

Rest of the paper is organized as follows. In section 2, the fourth order scheme is developed and its convergence analysis is studied. In section 3, the computational efficiency of new method is discussed and is compared with the methods which lie in the same category. Various numerical examples are considered in section 4 to show the consistent convergence behavior of the method and to verify the theoretical results. Section 5 contains the concluding remarks.

2. DEVELOPMENT OF THE METHOD

As stated above an efficient iterative method is one which possesses a higher convergence order with minimum computational cost. The most obvious barrier in the development of an efficient iterative scheme for solving systems of nonlinear equations is the evaluation of inverse of a matrix since it requires a lengthy and cumbersome computational work. Therefore, while constructing a numerical algorithm it will be more appropriate if the number of matrix inversions is as small as possible. Keeping this in mind we consider the following scheme:

$$\begin{aligned} y^{(k)} &= \mathrm{M}_{2,1}(x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - (a\,I + [w^{(k)}, x^{(k)}\,; F]^{-1}(b\,[y^{(k)}, x^{(k)}\,; F] + c\,[y^{(k)}, w^{(k)}\,; F])) \\ &\quad \times\, [w^{(k)}, x^{(k)}\,; F]^{-1} F(y^{(k)}), \end{aligned} \tag{4}$$

where $\mathrm{M}_{2,1}(x^{(k)})$ is the iterative scheme defined by (1), I is the identity matrix and a, b, c are some parameters to be determined.

In order to find the convergence order of scheme (4) we first define the divided difference operator for multivariable function F (see [**5**]). The divided difference operator of F is a mapping $[\cdot, \cdot\,; F] : D \times D \subset R^m \times R^m \to L(R^m)$ defined by

$$[x+h, x\,; F] = \int_0^1 F'(x+th)\,\mathrm{d}t,\ \forall\, x, h\ \in R^m. \tag{5}$$

Expanding $F'(x+th)$ in Taylor series around the point x and integrating, we have

$$[x+h, x\,; F] = \int_0^1 F'(x+th)\,\mathrm{d}t = F'(x) + \frac{1}{2}F''(x)h + \frac{1}{6}F'''(x)h^2 + O(h^3), \tag{6}$$

where $h^i = (h, h, \overset{i}{\cdots}, h)$, $h \in R^m$.
Let $e^{(k)} = x^{(k)} - \alpha$. Assuming that $\Gamma = [F'(\alpha)]^{-1}$ exists, then, developing $F(x^{(k)})$ and its first three derivatives around α, we have

$$F(x^{(k)}) = F'(\alpha)(e^{(k)} + A_2(e^{(k)})^2 + A_3(e^{(k)})^3 + A_4(e^{(k)})^4 + O((e^{(k)})^5)), \tag{7}$$

$$F'(x^{(k)}) = F'(\alpha)(I + 2A_2 e^{(k)} + 3A_3(e^{(k)})^2 + 4A_4(e^{(k)})^3 + O((e^{(k)})^4)), \tag{8}$$

$$F''(x^{(k)}) = F'(\alpha)(2A_2 + 6A_3 e^{(k)} + 12A_4(e^{(k)})^2 + O((e^{(k)})^3)) \tag{9}$$

and

$$F'''(x^{(k)}) = F'(\alpha)(6A_3 + 24A_4 e^{(k)} + O((e^{(k)})^2)), \tag{10}$$

where $A_i = \frac{1}{i!}\Gamma F^{(i)}(\alpha) \in L_i(R^m, R^m)$ and $(e^{(k)})^i = (e^{(k)}, e^{(k)}, \overset{i}{\cdots}, e^{(k)}, e^{(k)})$, $e^{(k)} \in R^m$.

We can now analyze the behavior of the scheme (4) through the following theorem.

Theorem 1. *Let the function $F : D \subset R^m \to R^m$ be sufficiently differentiable in an open neighborhood D of its solution α. If an initial approximation $x^{(0)}$ is sufficiently close to α, then the local order of convergence of method* (4) *is at least* 4 *provided,* $a = 3$, $b = -1$ *and* $c = -1$.

Proof. Let $\tilde{e}^{(k)} = w^{(k)} - \alpha$. Employing Eq. (6) for $x + h = w^{(k)}$, $x = x^{(k)}$, $h = \tilde{e}^{(k)} - e^{(k)}$ and then using (8)-(10), we have

$$[w^{(k)}, x^{(k)}; F] = F'(\alpha)(I + A_2(\tilde{e}^{(k)} + e^{(k)}) + A_3((\tilde{e}^{(k)})^2 + \tilde{e}^{(k)} e^{(k)} + (e^{(k)})^2) + O((e^{(k)})^3)).$$

Since $\tilde{e}^{(k)} = w^{(k)} - \alpha = x^{(k)} + \beta F(x^{(k)}) - \alpha = e^{(k)} + \beta F(x^{(k)}) = (I + \beta F'(\alpha))e^{(k)} + O((e^{(k)})^2)$. Thus, from the equation above it follows that

$$[w^{(k)}, x^{(k)}; F] = F'(\alpha)D(e^{(k)}) + O((e^{(k)})^3),$$

where $D(e^{(k)}) = I + A_2(2I + \beta F'(\alpha))e^{(k)} + A_3(3I + 3\beta F'(\alpha) + \beta^2(F'(\alpha))^2)(e^{(k)})^2$. Then, we obtain

$$[w^{(k)}, x^{(k)}; F]^{-1} = \big([D(e^{(k)})]^{-1} + O((e^{(k)})^3)\big)\Gamma. \tag{11}$$

The inverse $[D(e^{(k)})]^{-1}$ of $D(e^{(k)})$ is given by (see [**1, 2**])

$$[D(e^{(k)})]^{-1} = I + X_2 e^{(k)} + X_3 (e^{(k)})^2 + O((e^{(k)})^3), \tag{12}$$

where X_2 and X_3 satisfy the definition

$$D(e^{(k)})[D(e^{(k)})]^{-1} = [D(e^{(k)})]^{-1} D(e^{(k)}) = I. \tag{13}$$

Solving the system (13), we get

$$X_2 = -A_2(2I + \beta F'(\alpha))$$

and

$$X_3 = A_2^2(4I + 4\beta F'(\alpha) + \beta^2(F'(\alpha))^2) - A_3(3I + 3\beta F'(\alpha) + \beta^2(F'(\alpha))^2).$$

Thus, we find

$$\begin{aligned}[w^{(k)}, x^{(k)}; F]^{-1} = (I - A_2(2I + \beta F'(\alpha))e^{(k)} + (A_2^2(4I + 4\beta F'(\alpha) + \beta^2(F'(\alpha))^2) \\ - A_3(3I + 3\beta F'(\alpha) + \beta^2(F'(\alpha))^2))(e^{(k)})^2 + O((e^{(k)})^3))\Gamma.\end{aligned} \tag{14}$$

The first step of (4), using (7) and (14) to requisite terms, yields

$$\begin{aligned} E^{(k)} = y^{(k)} - \alpha &= e^{(k)} - [w^{(k)}, x^{(k)}; F]^{-1} F(x^{(k)}) \\ &= A_2(I + \beta F'(\alpha))(e^{(k)})^2 + O((e^{(k)})^3). \end{aligned} \tag{15}$$

Employing (6) for $x+h=y^{(k)}$, $x=x^{(k)}$ and $h=E^{(k)}-e^{(k)}$, it follows that

$$[y^{(k)},x^{(k)};F]=F'(\alpha)(I+A_2(E^{(k)}+e^{(k)})+A_3(e^{(k)})^2+O((e^{(k)})^3)). \tag{16}$$

Similarly, for $x+h=y^{(k)}$, $x=w^{(k)}$, $h=E^{(k)}-\tilde{e}^{(k)}$, Eq. (6) gives

$$[y^{(k)},w^{(k)};F]=F'(\alpha)(I+A_2(E^{(k)}+\tilde{e}^{(k)})+A_3(\tilde{e}^{(k)})^2+O((e^{(k)})^3)). \tag{17}$$

Then, from Eqs. (14), (16) and (17)

$$\begin{aligned}&aI+[w^{(k)},x^{(k)};F]^{-1}(b\,[y^{(k)},x^{(k)};F]+c\,[y^{(k)},w^{(k)};F])\\&=(a+b+c)I-A_2((b+c)I+b\beta F'(\alpha))e^{(k)}+(A_2^2(3(b+c)I\\&+(4b+2c)\beta F'(\alpha)+b\beta^2(F'(\alpha))^2)-A_3(2(b+c)I+(3b+c)\beta F'(\alpha)\\&+b\beta^2(F'(\alpha))^2))(e^{(k)})^2+O((e^{(k)})^3)).\end{aligned} \tag{18}$$

Post-multiplying Eq. (18) by $[w^{(k)},x^{(k)};F]^{-1}$ and simplifying

$$\begin{aligned}\theta&=(aI+[w^{(k)},x^{(k)};F]^{-1}(b\,[y^{(k)},x^{(k)};F]+c\,[y^{(k)},w^{(k)};F]))[w^{(k)},x^{(k)};F]^{-1}\\&=\big((a+b+c)I-A_2((2a+3(b+c))I+(a+2b+c)\beta F'(\alpha))e^{(k)}\\&+(A_2^2((4a+9(b+c))I+(4a+11b+7c)\beta F'(\alpha)+(a+3b+c)\beta^2(F'(\alpha))^2)\\&-A_3((3a+5(b+c))I+(3a+6b+4c)\beta F'(\alpha)+(a+2b+c)\beta^2(F'(\alpha))^2))(e^{(k)})^2\\&+O((e^{(k)})^3)\big)\Gamma.\end{aligned} \tag{19}$$

Substituting (19) and Taylor series of $F(y^{(k)})$ in the second step of (4), then using (15), we obtain

$$\begin{aligned}e^{(k+1)}&=x^{(k+1)}-\alpha=E^{(k)}-\theta F'(\alpha)(E^{(k)}+A_2(E^{(k)})^2+O((E^{(k)})^3))\\&=-B_1A_2(I+\beta F'(\alpha))(e^{(k)})^2+A_2^2(B_2I+B_3\beta F'(\alpha)+B_4\beta^2(F'(\alpha))^2)(e^{(k)})^3\\&+(A_3A_2(B_5I+B_6\beta F'(\alpha)+B_7\beta^2(F'(\alpha))^2+B_4\beta^3(F'(\alpha))^3)\\&-A_2^3(5B_8I+2B_9\beta F'(\alpha)+B_{10}\beta^2(F'(\alpha))^2+B_{11}\beta^3(F'(\alpha))^3))(e^{(k)})^4+O((e^{(k)})^5).\end{aligned} \tag{20}$$

where

$$\begin{aligned}&B_1=a+b+c-1,\quad B_2=2a+3b+3c,\quad B_3=3a+5b+4c,\quad B_4=a+2b+c,\\&B_5=3a+5b+5c,\quad B_6=6a+11b+9c,\quad B_7=4a+8b+5c,\quad B_8=a+2b+2c,\\&B_9=5a+11b+9c,\quad B_{10}=6a+15b+9c\quad\text{and}\quad B_{11}=a+3b+c.\end{aligned}$$

Our aim is to find values of the parameters a,b and c in such a way that the proposed iterative scheme (4) may produce order of convergence as high as possible. Thus, it will suffice to equate the coefficients B_i ($i=1$ to 3) to zero. Thus, solving

$$a+b+c=1,\quad 2a+3b+3c=0,\quad 3a+5b+4c=0,$$

we get $a = 3$, $b = -1$, $c = -1$. This set of values also satisfies $B_4 = 0$. Therefore, the error equation (20) reduces to

$$\begin{aligned} e^{(k+1)} &= (A_2^2(5I + 10\beta F'(\alpha) + 6\beta^2(F'(\alpha))^2 + \beta^3(F'(\alpha))^3) \\ &\quad - A_3(I + 2\beta F'(\alpha) + \beta^2(F'(\alpha))^2))A_2(e^{(k)})^4 + O((e^{(k)})^5). \end{aligned} \tag{21}$$

This completes the proof of theorem 1. □

Finally, the fourth order scheme (4) is given by

$$\begin{aligned} y^{(k)} &= \mathrm{M}_{2,1}(x^{(k)}), \\ x^{(k+1)} &= \mathrm{M}_{4,3}(x^{(k)}, y^{(k)}) = y^{(k)} - (3\,I - [w^{(k)}, x^{(k)}\,; F]^{-1} \\ &\quad \times ([y^{(k)}, x^{(k)}\,; F] + [y^{(k)}, w^{(k)}\,; F]))[w^{(k)}, x^{(k)}\,; F]^{-1}F(y^{(k)}). \end{aligned} \tag{22}$$

3. COMPUTATIONAL EFFICIENCY

To obtain an estimation of the efficiency of proposed method we use the efficiency index. The efficiency of an iterative method is given by $\mathrm{E} = p^{\frac{1}{C}}$ (see [**11**]), where p is the order of convergence and C is the computational cost per iteration. For a system of m nonlinear equations with m variables, the computational cost per iteration is given by (see [**6**])

$$C(\mu, m, \ell) = A(m)\mu + P(m, \ell), \tag{23}$$

where $A(m)$ denotes the number of evaluations of scalar functions used in the evaluation of F and $[x, y\,; F]$, and $P(m, \ell)$ denotes the number of products needed per iteration. The divided difference $[x, y\,; F]$ of F is an $m \times m$ matrix with elements (see [**5, 16**])

$$[x, y\,; F]_{ij} = \frac{f_i(x_1, \ldots, x_j, y_{j+1}, \ldots, y_m) - f_i(x_1, \ldots, x_{j-1}, y_j, \ldots, y_m)}{x_j - y_j}, \quad 1 \le i,\, j \le m.$$

In order to express the value of $C(\mu, m, \ell)$ in terms of products, a ratio $\mu > 0$ between products and evaluations of scalar functions, and a ratio $\ell \geqslant 1$ between products and quotients is required.

To compute F in any iterative function we evaluate m scalar functions $(f_1, f_2, \ldots, f_m)$ and if we compute a divided difference $[x, y\,; F]$ then we evaluate $m(m-1)$ scalar functions, where $F(x)$ and $F(y)$ are computed separately. We must add m^2 quotients from any divided difference and m products for multiplication of a vector by a scalar. In order to compute an inverse linear operator we solve a linear system, where we have $m(m-1)(2m-1)/6$ products and $m(m-1)/2$ quotients in the LU decomposition, and $m(m-1)$ products and m quotients in the resolution of two triangular linear systems.

We compare the computational efficiency of present fourth order method $\mathrm{M}_{4,3}$ with existing second order method $\mathrm{M}_{2,1}$, fourth order methods $\mathrm{M}_{4,1}$ and $\mathrm{M}_{4,2}$, and seventh order methods $\mathrm{M}_{7,1}$ and $\mathrm{M}_{7,2}$. Let us denote efficiency indices of $\mathrm{M}_{p,i}$ by

$E_{p,i}$ and computational cost by $C_{p,i}$. Taking into account the above considerations, we have

(24) $C_{2,1} = \frac{m}{6}(2m^2 + 6m\mu + 3m + 9\ell m + 6\mu + 3\ell - 5)$ and $E_{2,1} = 2^{1/C_{2,1}}$.

(25) $C_{4,1} = \frac{m}{3}(2m^2 + 9m\mu + 3m + 12\ell m + 3\ell - 5)$ and $E_{4,1} = 4^{1/C_{4,1}}$.

(26) $C_{4,2} = \frac{m}{3}(2m^2 + 9m\mu + 6m + 12\ell m + 6\ell - 8)$ and $E_{4,2} = 4^{1/C_{4,2}}$.

(27) $C_{4,3} = \frac{m}{6}(2m^2 + 18m\mu + 15m + 21\ell m + 15\ell - 11)$ and $E_{4,3} = 4^{1/C_{4,3}}$.

(28) $C_{7,1} = \frac{m}{2}(2m^2 + 10m\mu + 3m + 13\ell m - 2\mu + 3\ell - 5)$ and $E_{7,1} = 7^{1/C_{7,1}}$.

(29) $C_{7,2} = \frac{m}{2}(2m^2 + 10m\mu + 5m + 13\ell m - 2\mu + 5\ell - 7)$ and $E_{7,2} = 7^{1/C_{7,2}}$.

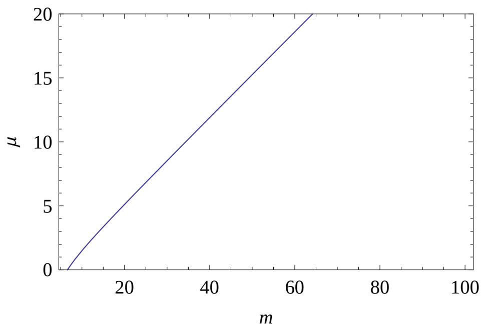

Figure 1. Boundary $R_{4,3;2,1} = 1$ in (m, μ)-plane for $\ell = 1$.

3.1 Comparison Between Efficiencies

To compare the iterative methods $M_{p,i}$, we consider the ratio

$$R_{p,i;q,j} = \frac{\log E_{p,i}}{\log E_{q,j}} = \frac{C_{q,j}\ \log(p)}{C_{p,i}\ \log(q)}. \tag{30}$$

It is clear that if $R_{p,i;q,j} > 1$, the iterative method $M_{p,i}$ is more efficient than $M_{q,j}$. Taking into account that the border between two computational efficiencies is given by $R_{p,i;q,j} = 1$, this boundary is given by the equation μ written as a function of ℓ and m; $\mu > 0$, m is a positive integer $\geqslant 2$ and $\ell \geqslant 1$.

$M_{4,3}$ *versus* $M_{2,1}$ *case*:

For this case the ratio (30) is given by

$$R_{4,3;2,1} = \frac{2(2m^2 + 6m\mu + 3m + 9\ell m + 6\mu + 3\ell - 5)}{2m^2 + 18m\mu + 15m + 21\ell m + 15\ell - 11}.$$

The boundary $R_{4,3;2,1} = 1$, expressed by μ written as a function of ℓ and m, is

$$\mu = \frac{2m^2 - 3\ell(m+3) - 9m + 1}{6(m-2)}. \tag{31}$$

The function μ has one vertical asymptote for $m = 2$ and for all $\ell \geqslant 1$. Note that the denominator of (31) is positive for $m > 2$ and the numerator is negative for $m = 2, \ldots, 6$, which shows that μ is always negative for $m = 2, \ldots, 6$. That is, the boundary is out of admissible region and as a consequence for $2 \leqslant m \leqslant 6$, we have $\mathrm{E}_{4,3} < \mathrm{E}_{2,1}$, $\forall \mu > 0$ and $\ell \geqslant 1$. For $m > 6$ and a fixed value of ℓ, the boundary (31) divides the efficiency region between $\mathrm{M}_{4,3}$ and $\mathrm{M}_{2,1}$ in (m, μ)-plane. In particular, for $\ell = 1$, the boundary is shown in Fig. 1, where $\mathrm{E}_{4,3} > \mathrm{E}_{2,1}$ on the right and $\mathrm{E}_{4,3} < \mathrm{E}_{2,1}$ on the left of the boundary.

$\mathrm{M}_{4,3}$ *versus* $\mathrm{M}_{4,1}$ *case*:

$$R_{4,3;4,1} = \frac{2(2m^2 + 9m\mu + 3m + 12\ell m + 3\ell - 5)}{2m^2 + 18m\mu + 15m + 21\ell m + 15\ell - 11}.$$

In this case it is easy to prove that $R_{4,3;4,1} > 1$, $\forall\ \mu > 0$, $\ell \geqslant 1$ and $m \geqslant 5$. Thus, we conclude that $\mathrm{E}_{4,3} > \mathrm{E}_{4,1}$ for $\mu > 0$, $\ell \geqslant 1$ and $m \geqslant 5$. Also, for $m = 4$, $\ell > 1$ we have $R_{4,3;4,1} > 1$ which implies that $\mathrm{E}_{4,3} > \mathrm{E}_{4,1}$ and for $m = 4$, $\ell = 1$ we have $R_{4,3;4,1} = 1$ which gives $\mathrm{E}_{4,3} = \mathrm{E}_{4,1}$.

$\mathrm{M}_{4,3}$ *versus* $\mathrm{M}_{4,2}$ *case*:

$$R_{4,3;4,2} = \frac{2(2m^2 + 9m\mu + 6m + 12\ell m + 6\ell - 8)}{2m^2 + 18m\mu + 15m + 21\ell m + 15\ell - 11}.$$

With the same values of μ, ℓ as in previous cases and $\forall\ m \geqslant 3$ the ratio $R_{4,3;4,2} > 1$, which implies that $\mathrm{E}_{4,3} > \mathrm{E}_{4,2}$. Also, for $m = 2$, $\ell > 1$ we have $R_{4,3;4,2} > 1$ which gives $\mathrm{E}_{4,3} > \mathrm{E}_{4,2}$ and for $m = 2$, $\ell = 1$ we have $R_{4,3;4,2} = 1$ which means that $\mathrm{E}_{4,3} = \mathrm{E}_{4,2}$.

$\mathrm{M}_{4,3}$ *versus* $\mathrm{M}_{7,1}$ *case*:

$$R_{4,3;7,1} = \frac{3(2m^2 + 10m\mu + 3m + 13\ell m - 2\mu + 3\ell - 5)\log(4)}{(2m^2 + 18m\mu + 15m + 21\ell m + 15\ell - 11)\log(7)}.$$

Here, $\forall\ \mu > 0$ and $\ell \geqslant 1$ we have $R_{4,3;7,1} > 1$ for $m \geqslant 3$, which gives that $\mathrm{E}_{4,3} > \mathrm{E}_{7,1}$ for $m \geqslant 3$.

$\mathrm{M}_{4,3}$ *versus* $\mathrm{M}_{7,2}$ *case*:

$$R_{4,3;7,2} = \frac{3(2m^2 + 10m\mu + 5m + 13\ell m - 2\mu + 5\ell - 7)\log(4)}{(2m^2 + 18m\mu + 15m + 21\ell m + 15\ell - 11)\log(7)}.$$

In this case $\forall\ \mu > 0$ and $\ell \geqslant 1$ we have $R_{4,3;7,2} > 1$ for $m \geqslant 2$, which means that $\mathrm{E}_{4,3} > \mathrm{E}_{7,2}$ for $m \geqslant 2$.

We summarize the above results in the following theorem.

Theorem 2. *For all $\mu > 0$ and $\ell \geqslant 1$ we have:*

(i) $\mathrm{E}_{4,3} < \mathrm{E}_{2,1}$ *for* $2 \leqslant m \leqslant 6$.
(ii) $\mathrm{E}_{4,3} \geqslant \mathrm{E}_{4,1}$ *for* $m \geqslant 4$.
(iii) $\mathrm{E}_{4,3} \geqslant \mathrm{E}_{4,2}$ *and* $\mathrm{E}_{4,3} > \mathrm{E}_{7,2}$ *for* $m \geqslant 2$.

(iv) $\mathrm{E}_{4,3} > \mathrm{E}_{7,1}$ *for* $m \geqslant 3$.

Otherwise, the comparison depends on m, μ *and* ℓ.

In order to verify the results of Theorem 2 we plot graphs for the set $(\mu, \ell) = (1, 1)$. These graphs in (m, E)-variables are shown in Figures 2-5.

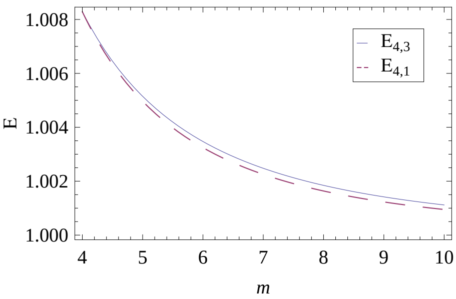

Figure 2. Plots for E values of $\mathrm{M}_{4,3}$ and $\mathrm{M}_{4,1}$ for $m \geqslant 4$.

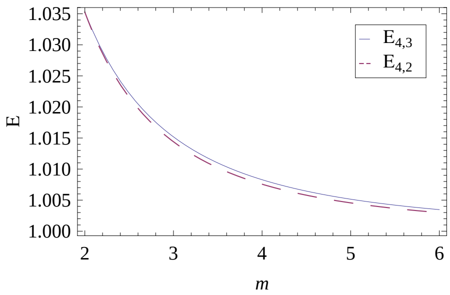

Figure 3. Plots for E values of $\mathrm{M}_{4,3}$ and $\mathrm{M}_{4,2}$ for $m \geqslant 2$.

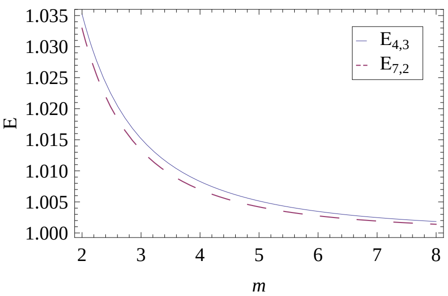

Figure 4. Plots for E values of $\mathrm{M}_{4,3}$ and $\mathrm{M}_{7,2}$ for $m \geqslant 2$.

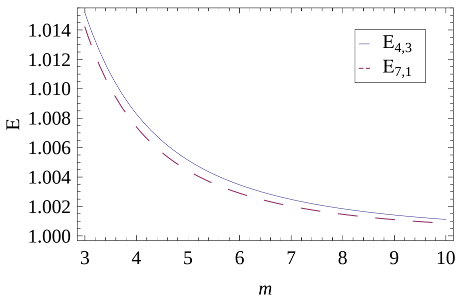

Figure 5. Plots for E values of $\mathrm{M}_{4,3}$ and $\mathrm{M}_{7,1}$ for $m \geqslant 3$.

4. NUMERICAL RESULTS

In this section, some numerical problems are considered to illustrate the convergence behavior and computational efficiency of the proposed method. A comparison with the existing methods $\mathrm{M}_{2,1}$, $\mathrm{M}_{4,1}$, $\mathrm{M}_{4,2}$, $\mathrm{M}_{7,1}$ and $\mathrm{M}_{7,2}$ is also presented. All computations are performed in *Mathematica* [**22**] using multiple-precision arithmetic with 2048 digits. To confirm the theoretical order of convergence, we calculate the computational order of convergence (p_c) using the formula [**12**]

$$p_c = \frac{\log(||F(x^{(k)})||/||F(x^{(k-1)})||)}{\log(||F(x^{(k-1)})||/||F(x^{(k-2)})||)},$$

taking into consideration the last three approximations in the iterative process.

According to the definition of the computational cost (23), an estimation of the factor μ is claimed. In order to do this, we express the cost of the evaluation of the elementary functions in terms of products, which depends on the computer, the software and the arithmetics used [**4, 7**]. In Table 1, an estimation of the cost of the elementary functions in product units is displayed, wherein the running time of one product is measured in milliseconds. For the hardware and the software used in the numerical work, the computational cost of quotient with respect to product is $\ell = 3$ (see Table 1).

Digits	$x * y$	x/y	$\sqrt{x}$	$\exp(x)$	$\ln(x)$	$\sin(x)$	$\cos(x)$	$\arccos(x)$	$\arctan(x)$
2048	0.0301 ms	3	1.5	77	78	77	77	119	118

Table 1. Estimation of computational cost of elementary functions computed with Mathematica 7.0 in a processor Intel (R) Core (TM) i5-2430M CPU @ 2.40 GHz (32-bit Machine) Microsoft Windows 7 Ultimate 2009, where $x = \sqrt{3} - 1$ and $y = \sqrt{5}$.

The Traub's method $M_{2,1}$ and the present method $M_{4,3}$ are tested by using the values -0.01 and 0.01 for the parameter β. The following problems are chosen for numerical tests:

Problem 1. Considering the system of two equations (see [**21**]):

$$(x-1)^4 + e^{-y} - y^2 + 3y + 1 = 0,$$
$$4\sin(x-1) - \log(x^2 - x + 1) - y^2 = 0.$$

In this problem $(m, \mu) = (2, 120)$ are the values used in (24)-(29) for calculating computational costs and efficiency indices of the methods. The initial approximation chosen is $x^{(0)} = \{2, -2\}^t$ and the solution
$\alpha = \{2.0704433766798807\ldots, -1.5301712023005783\ldots\}^t$.

Problem 2. Now considering the system of five equations (selected from [**6**]):

$$\sum_{j=1, j\neq i}^{5} x_j - e^{-x_i} = 0, \quad 1 \leq i \leq 5,$$

with initial value $x^{(0)} = \{1, 1, 1, 1, 1\}^t$. Solution of this problem is,
$\alpha = \{0.20388835470224016\ldots, 0.20388835470224016\ldots, 0.20388835470224016\ldots,$
$0.20388835470224016\ldots, 0.20388835470224016\ldots\}^t$. The concrete values of parameters used in (24)-(29) are $(m, \mu) = (5, 77)$.

Problem 3. Next, Considering the mixed Hammerstein integral equation (see [**10**]):

$$x(s) = 1 + \frac{1}{5}\int_0^1 G(s,t)x(t)^3 \mathrm{d}t,$$

where $x \in C[0,1]$; $s, t \in [0,1]$ and the kernel G is

$$G(s,t) = \begin{cases} (1-s)t, & t \leqslant s, \\ s(1-t), & s \leqslant t. \end{cases}$$

We transform the above equation into a finite-dimensional problem by using Gauss-Legendre quadrature formula given as

$$\int_0^1 f(t)\,\mathrm{d}t \approx \sum_{j=1}^{m} \varpi_j f(t_j),$$

where the abscissas t_j and the weights ϖ_j are determined for $m = 8$ by Gauss-Legendre quadrature formula. Denoting the approximation of $x(t_i)$ by x_i $(i = 1, 2, \ldots, 8)$, we obtain the system of nonlinear equations

$$5x_i - 5 - \sum_{j=1}^{8} a_{ij} x_j^3 = 0,$$

where, for $i = 1, 2, \ldots, 8$,

$$a_{ij} = \begin{cases} \varpi_j t_j (1 - t_i) & \text{if } j \leqslant i, \\ \varpi_j t_i (1 - t_j) & \text{if } i < j, \end{cases}$$

wherein the abscissas t_j and the weights ϖ_j are known and given in Table 2 for $m = 8$. In this case the concrete values of parameters are $(m, \mu) = (8, 11)$, which we use in (24)-(29) for the calculation of costs and efficiency indices of the methods. Initial approximation chosen for solving this problem is,

$$x^{(0)} = \{-0.5,\ -0.5,\ -0.5,\ -0.5,\ -0.5,\ -0.5,\ -0.5,\ -0.5\}^t,$$

towards the solution

$$\alpha = \{1.0020962450311568\ldots, 1.0099003161874888\ldots, 1.0197269609931769\ldots, 1.0264357430306205\ldots, 1.0264357430306205\ldots, 1.0197269609931769\ldots, 1.0099003161874888\ldots, 1.0020962450311568\ldots\}^t.$$

j	t_j	ϖ_j
1	0.0198550717512318 . . .	0.0506142681451881 . . .
2	0.1016667612931866 . . .	0.1111905172266872 . . .
3	0.2372337950418355 . . .	0.1568533229389436 . . .
4	0.4082826787521750 . . .	0.1813418916891809 . . .
5	0.5917173212478249 . . .	0.1813418916891809 . . .
6	0.7627662049581644 . . .	0.1568533229389436 . . .
7	0.8983332387068133 . . .	0.1111905172266872 . . .
8	0.9801449282487681 . . .	0.0506142681451881 . . .

Table 2. Abscissas and weights of Gauss-Legendre quadrature formula for $m = 8$.

Problem 4. Lastly, considering the system of twenty equations:

$$\begin{cases} x_i^2 x_{i+1} - 1 = 0, & 1 \le i \le 19, \\ x_{20}^2 x_1 - 1 = 0. \end{cases}$$

The initial approximation chosen is $x^{(0)} = \{1.5, 1.5, \ldots, 1.5\}^t$ for obtaining the solution $\alpha = \{1, 1, \ldots, 1\}^t$. Here the concrete values of parameters used in (24)-(29) are $(m, \mu) = (20, 2)$.

Methods	$\|\|x^{(1)}-\alpha\|\|$	$\|\|x^{(2)}-\alpha\|\|$	$\|\|x^{(3)}-\alpha\|\|$	p_c	C	E
$M_{2,1}(\beta=-.01)$	1.22(−1)	2.12(−2)	6.96(−4)	1.967	744	1.000932
$M_{2,1}(\beta=.01)$	1.29(−1)	2.67(−2)	1.21(−3)	1.955	744	1.000932
$M_{4,1}$	2.21(−1)	3.34(−2)	5.57(−5)	3.917	1500	1.000925
$M_{4,2}$	2.83(−1)	7.81(−2)	1.51(−3)	3.676	1508	1.000920
$M_{4,3}(\beta=-.01)$	3.31(−2)	1.60(−5)	1.12(−18)	4.000	1506	1.000921
$M_{4,3}(\beta=.01)$	4.20(−2)	4.55(−5)	8.40(−17)	4.000	1506	1.000921
$M_{7,1}$	7.12(−2)	3.49(−7)	7.44(−46)	6.944	2256	1.000863
$M_{7,2}$	1.21(−1)	2.90(−5)	1.59(−30)	7.008	2264	1.000860

Table 3. Performance of methods for Problem 1.

Methods	$\|\|x^{(1)}-\alpha\|\|$	$\|\|x^{(2)}-\alpha\|\|$	$\|\|x^{(3)}-\alpha\|\|$	p_c	C	E
$M_{2,1}(\beta=-.01)$	7.62(−2)	2.14(−4)	1.65(−9)	2.000	2480	1.000280
$M_{2,1}(\beta=.01)$	8.18(−2)	2.72(−4)	2.94(−9)	2.000	2480	1.000280
$M_{4,1}$	7.08(−3)	1.34(−11)	1.74(−46)	4.000	6190	1.000224
$M_{4,2}$	6.98(−3)	1.20(−11)	1.05(−46)	4.000	6225	1.000223
$M_{4,3}(\beta=-.01)$	1.10(−4)	7.55(−21)	1.71(−85)	4.000	6170	1.000225
$M_{4,3}(\beta=.01)$	1.45(−4)	2.81(−20)	3.97(−83)	4.000	6170	1.000225
$M_{7,1}$	1.06(−5)	1.01(−40)	7.32(−286)	7.000	9900	1.000197
$M_{7,2}$	1.05(−5)	8.81(−41)	2.60(−286)	7.000	9935	1.000196

Table 4. Performance of methods for Problem 2.

Methods	$\|\|x^{(1)}-\alpha\|\|$	$\|\|x^{(2)}-\alpha\|\|$	$\|\|x^{(3)}-\alpha\|\|$	p_c	C	E
$M_{2,1}(\beta=-.01)$	3.94(−3)	5.12(−7)	8.88(−15)	2.000	1288	1.000538
$M_{2,1}(\beta=.01)$	7.77(−4)	2.15(−8)	1.61(−17)	2.000	1288	1.000538
$M_{4,1}$	3.38(−2)	1.15(−9)	1.63(−39)	4.000	3296	1.000421
$M_{4,2}$	3.45(−2)	1.36(−9)	3.44(−39)	4.000	3376	1.000411
$M_{4,3}(\beta=-.01)$	1.91(−4)	5.06(−19)	2.62(−77)	4.000	3160	1.000439
$M_{4,3}(\beta=.01)$	3.87(−5)	8.98(−22)	2.70(−88)	4.000	3160	1.000439
$M_{7,1}$	2.17(−4)	7.88(−33)	7.07(−232)	7.000	5304	1.000367
$M_{7,2}$	2.22(−4)	1.00(−32)	4.05(−231)	7.000	5384	1.000361

Table 5. Performance of methods for Problem 3.

Methods	$\|\|x^{(1)}-\alpha\|\|$	$\|\|x^{(2)}-\alpha\|\|$	$\|\|x^{(3)}-\alpha\|\|$	p_c	C	E
$M_{2,1}(\beta=-.01)$	5.64(−1)	7.26(−2)	1.15(−3)	1.987	5520	1.000126
$M_{2,1}(\beta=.01)$	5.97(−1)	8.84(−2)	1.83(−3)	1.983	5520	1.000126
$M_{4,1}$	3.95(−1)	2.00(−3)	1.92(−12)	4.000	12960	1.000107
$M_{4,2}$	4.11(−1)	2.88(−3)	1.13(−11)	4.000	13400	1.000103
$M_{4,3}(\beta=-.01)$	1.65(−1)	2.98(−5)	3.88(−20)	4.000	10380	1.000134
$M_{4,3}(\beta=.01)$	1.86(−1)	5.27(−5)	4.26(−19)	4.000	10380	1.000134
$M_{7,1}$	1.80(−2)	1.01(−15)	1.88(−108)	7.000	20400	1.000095
$M_{7,2}$	2.04(−2)	3.31(−15)	1.06(−104)	7.000	20840	1.000093

Table 6. Performance of methods for Problem 4.

The errors $||x^{(k)}-\alpha||$ of approximations to the corresponding solutions of problems 1-4, the computational order of convergence (p_c), the computational costs $C_{p,i}$ given by (24)-(29) in terms of products and the corresponding computational efficiencies $E_{p,i}$ are displayed in Tables 3-6, where $b(-a)$ denotes $b\times 10^{-a}$. From the results displayed in Tables 3-6 it is clear that the accuracy in numerical values of approximations to the solution increases as the iteration process proceeds, showing stable nature of the methods. It can also be observed that the fourth order method shows robust character in terms of accuracy when compared with the methods of inferior and same order. The seventh order methods produce approximations of greater accuracy due to their higher order of convergence, but possess less efficiency than the new method. Calculated values of the computational order of convergence displayed in the fifth column of Tables 3-6 verify the theoretical order of convergence proved in Section 2. Numerical values of the efficiency index (E) displayed in the last column of each table also confirm the theoretical results as stated in Theorem 2.

5. CONCLUSIONS

In the foregoing study, we have proposed an iterative method with fourth order of convergence for solving systems of nonlinear equations. The scheme is completely derivative free and therefore particularly suited to those problems in which derivatives require lengthy computation. A development of first-order divided difference operator for functions of several variables and direct computation by Taylor's expansion are used to prove the local convergence order of new method. A comparison of efficiencies of the new scheme with existing schemes is shown. It is observed that for large systems the present method has an edge over similar existing methods. Some numerical examples have been presented and the performance is compared with existing methods. Computational results have confirmed robust and efficient character of the proposed technique. Similar numerical experimentations have been carried out for a number of problems and results are found to be on a par with those presented here.

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REFERENCES

1. A. Cordero, J. L. Hueso, E. Martínez, J. R.Torregrosa: *A modified Newton-Jarratt's composition.* Numer. Algor., **55** (2010), 87–99.
2. A. Cordero, J. L. Hueso, E. Martínez, J. R.Torregrosa: *Efficient high-order methods based on golden ratio for nonlinear systems.* Appl. Math. Comput., **217** (2011), 4548–4556.
3. J. Džunić, M. S. Petković: *On generalized multipoint root-solvers with memory.* J. Comput. Appl. Math., **236** (2012), 2909–2920.
4. L. Fousse, G. Hanrot, V. Lefèvre, P. Pélissier, P. Zimmermann: *MPFR: a multiple-precision binary floating-point library with correct rounding.* ACM Trans. Math. Softw., **33** (2) (2007), 15. Art. 13.

5. M. Grau-Sánchez, À. Grau, M. Noguera: *Frozen divided difference scheme for solving systems of nonlinear equations.* J. Comput. Appl. Math., **235** (2011), 1739-1743.
6. M. Grau-Sánchez, M. Noguera: *A technique to choose the most efficient method between secant method and some variants.* Appl. Math. Comput., **218** (2012), 6415-6426.
7. http://www.mpfr.org/mpfr-2.1.0/timings.html.
8. C. T. Kelley: *Solving nonlinear equations with Newton's Method.* SIAM, Philadelphia, 2003.
9. Z. Liu, Q. Zheng, P. Zhao: *A variant of Steffensen's method of fourth-order convergence and its applications.* Appl. Math. Comput., **216** (2010), 1978–1983.
10. J. M. Ortega, W. C. Rheinboldt: *Iterative Solution of Nonlinear Equations in Several Variables.* Academic Press, New York, 1970.
11. A. M. Ostrowski: *Solution of Equations and System of Equations.* Academic Press, New York, 1960.
12. M. S. Petković: *Remarks on "On a general class of multipoint root-finding methods of high computational efficiency".* SIAM J. Numer. Anal., **49** (2011), 1317–1319.
13. M. S. Petković, L. D. Petković: *Families of optimal multipoint methods for solving nonlinear equations: a survey.* Appl. Anal. Discrete Math., **4** (2010), 1–22.
14. M. S. Petković, J. Džunić, L. D. Petković: *A family of two-point methods with memory for solving nonlinear equations.* Appl. Anal. Discrete Math., **5** (2011), 298–317.
15. M. S. Petković, S. Ilić, J. Džunić: *Derivative free two-point methods with and without memory for solving nonlinear equations.* Appl. Math. Comput., **217** (2010), 1887–1895.
16. F-A. Potra, V. Pták: *Nondiscrete Induction and Iterarive Processes.* Pitman Publishing, Boston, 1984.
17. H. Ren, Q. Wu, W. Bi: *A class of two-step Steffensen type methods with fourth-order convergence.* Appl. Math. Comput., **209** (2009), 206–210.
18. J. R. Sharma, R. K. Guha, P. Gupta: *Some efficient derivative free methods with memory for solving nonlinear equations.* Appl. Math. Comput., **219** (2012), 699–707.
19. J. F. Steffensen: *Remarks on iteration.* Skand. Aktuar Tidskr., **16** (1933), 64–72.
20. J. F. Traub: *Iterative Methods for the Solution of Equations.* Prentice-Hall, New Jersey, 1964.
21. X. Wang, T. Zhang: *A family of Steffensen type methods with seventh-order convergence.* Numer. Algor., **62** (2013), 229–444.
22. S. Wolfram: *The Mathematica Book, fifth ed.* Wolfram Media, 2003.
23. Q. Zheng, J. Li, F. Huang: *An optimal Steffensen-type family for solving nonlinear equations.* Appl. Math. Comput., **217** (2011), 9592–9597.

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