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TWO TYPES OF STABILITY CONDITIONS FOR LINEAR DELAY DIFFERENCE EQUATIONS

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The paper discusses asymptotic stability conditions for a four-parameter linear difference equation appearing in the process of discretization of a delay differential equation. We present two types of conditions, which are necessary and sufficient for asymptotic stability of the studied equation. A relationship between both the types of conditions is established and some of their consequences are discussed.

1. INTRODUCTION

This paper discusses asymptotic stability conditions for a linear difference equation

$$y(n+2) + \alpha\, y(n) + \beta\, y(n-k+2) + \gamma\, y(n-k) = 0, \qquad n = 0, 1, 2, \dots \tag{1}$$

with real constant entries α, β, γ and a positive integer $k > 2$. Stability issues traditionally belong among the key and frequently discussed topics of qualitative analysis of difference equations. In particular, the knowledge of necessary and sufficient stability conditions is of a special importance in these investigations. The basic theory of linear difference equations implies the equivalent formulation of this stability problem for (1), namely whether the associated characteristic polynomial

$$p(\lambda) \equiv \lambda^{k+2} + \alpha\, \lambda^k + \beta\, \lambda^2 + \gamma \tag{2}$$

is of a Schur type, i.e. all its zeros are located inside the unit circle. This is a classical polynomial problem answerable by use of appropriate criterions which

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are due to Cohn, Jury and Schur (see [**9, 15**]). These criterions work for the full-term polynomial

$$q(\lambda) \equiv \lambda^{k+1} + \sum_{j=0}^{k} a_j \lambda^{k-j}\,, \tag{3}$$

even with complex entries a_j $(j = 0, 1, \dots, k)$. However, they are effective only for $q(\lambda)$ with fixed k and fixed a_j in the sense, that they do not provide with general conditions in terms of k and a_j. This is true also for special polynomials $q(\lambda)$ involving only a few nonzero terms. A typical example of such a polynomial is

$$\widetilde{p}(\lambda) \equiv \lambda^{k+1} + \alpha\,\lambda^k + \beta\,\lambda + \gamma \tag{4}$$

representing the characteristic polynomial for the difference equation

$$y(n+1) + \alpha\, y(n) + \beta\, y(n-k+1) + \gamma\, y(n-k) = 0, \qquad n = 0, 1, 2, \dots\,. \tag{5}$$

Equations of this type are usually referred to as delay difference equations. Note that (5) was considered in the purely delayed case ($\beta = 0$), the advanced case ($\alpha = 0$), as well as in the mixed case ($\alpha\beta \neq 0$), and its corresponding asymptotic stability properties were reported in [**13**], [**6**] and [**2**], respectively. These properties can be easily extended to (1) with k even. More precisely, in such a case it is enough to replace λ^2 by λ and $k/2$ by k to obtain the characteristic polynomial (2) in the form (4). On this account, throughout this paper we investigate (1) only with k odd.

There are several reasons why to investigate equations (1) and (5). Firstly, they represent significant numerical discretizations of delay differential equations. In particular, if we consider the standard delay test equation

$$x'(t) + a\,x(t) + b\,x(t-\tau) = 0, \qquad a, b \in \mathbb{R}, \quad \tau \in \mathbb{R}^+\,, \tag{6}$$

then basic numerical formulae (such as Euler discretizations or the trapezoidal rule) have the form (5) with appropriate α, β, γ and k depending on a, b and τ (see [**1**]). Similarly, if we apply the midpoint rule to (6), then we arrive just at the formula (1), whose stability analysis is the main goal of this paper. Secondly, difference equations of these types appear in the study of various discrete problems, connected especially with population modeling or chaos. They appear also as linearized forms for (local) stability investigations of some nonlinear discrete models (see, e.g. [**8**] and [**14**]). The last motivation is purely theoretic and it is connected with the above stated (equivalent) polynomial problem. As noted by many authors, formulation of explicit necessary and sufficient conditions guaranteeing that a general polynomial (3) is of the Schur type, represents an extremely complicated matter, which seems to lie beyond theoretical possibilities (see, e.g. [**10**]). From this viewpoint, any extension of the known results to more general polynomials can be useful for a better understanding of the problem structure.

The organization of this paper is following. Section 2 presents a very brief overview of the known asymptotic stability conditions for a particular case of (1).

These conditions are of different types, which seem to be uneasy to compare. In Section 3, we extend these stability conditions to the case of (1). Their proofs along with some related comments are the subject of Section 4. Discussions of the derived stability criterions (including their mutual relationship and illustrative examples) are performed in Section 5. Several final remarks in the last section conclude the paper.

2. PRELIMINARIES

As it is customary, by asymptotic stability of the difference equation

$$y(n+1) + \sum_{j=0}^{k} a_j y(n-j) = 0, \qquad n = 0, 1, 2, \dots \tag{7}$$

we understand the property that $y(n)$ is tending to zero as $n \to \infty$ for any solution $y(n)$ of (7). Applying the Rouché's Theorem one can easily verify that the sufficient condition for asymptotic stability of (7) is

$$\sum_{j=0}^{k} |a_j| < 1 \tag{8}$$

(see [**12**]). This condition is called the Cohn stability domain and provides a direct generalization of the well-known Clark's condition for a three-term linear difference equation (see [**4**]).

In the case of (1), the notion of asymptotic stability implicitly assumes that the value k is fixed. To emphasize this fact, such a kind of stability is sometimes referred to as delay-dependent stability. In other words, for a given $k \in \mathbb{Z}^+$, $k > 2$ we can introduce the symbol

$$S(k) = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \lim_{n\to\infty} y(n) = 0 \text{ for any solution } y(n) \text{ of (1)}\}$$

and call it the delay-dependent asymptotic stability region for (1). If the definition limit property is independent of k, then we define

$$S = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \lim_{n\to\infty} y(n) = 0 \text{ for any solution } y(n) \text{ of (1) and any } k > 2\}$$

and call it the delay-independent asymptotic stability region for (1). Since the corresponding Cohn stability domain does not depend on k, using the notation

$$C = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : |\alpha| + |\beta| + |\gamma| < 1\}$$

one immediately gets $C \subset S$. On the other hand, it may occur $C = S = S(k)$ if we restrict the introductions of these stability sets to a special part of $\mathbb{R}^3$. Thus, one of our aims is to describe S and $S(k)$ for arbitrary real values of α, β, γ.

If $\beta = 0$ then two systems of necessary and sufficient conditions guaranteeing asymptotic stability of

$$y(n+2) + \alpha\, y(n) + \gamma\, y(n-k) = 0, \qquad n = 0, 1, 2, \ldots \tag{9}$$

are known. The following system is due to REN [**18**].

Theorem 2.1. *Let α, γ be nonzero real constants and k be a positive odd integer. Then (9) is asymptotically stable if and only if either*

$$-1 < \alpha < 0, \qquad |\gamma| < 1 + \alpha\,,$$

or

$$0 < \alpha < 1, \qquad |\gamma| < \left(\alpha^2 + 2\alpha \cos 2\phi + 1\right)^{1/2}\,,$$

where ϕ is the solution in the interval $((k+1)\pi/(2k+4), \pi/2)$ of the equation

$$\sin(kx)/\sin\left((k+2)x\right) = -1/\alpha\,. \tag{10}$$

Recently, two of the authors of this paper proved this criterion (see [**3**]).

Theorem 2.2. *Let α, γ be nonzero real constants and k be a positive odd integer.*

(i) *Let $\alpha < 0$. Then (9) is asymptotically stable if and only if*

$$|\alpha| + |\gamma| < 1\,.$$

(ii) *Let $\alpha > 0$. Then (9) is asymptotically stable if and only if either*

$$|\alpha| + |\gamma| \le 1\,,$$

or

$$\gamma^2 < 1 - \alpha < |\gamma|, \quad k < 2\left(\arcsin \frac{1-\alpha^2-\gamma^2}{2|\alpha\gamma|}\right)\Big/\left(\arccos \frac{\alpha^2-\gamma^2+1}{2|\alpha|}\right).$$

Our aim is to extend both the criterions to the case of (1) and thus explicitly describe the stability sets S and $S(k)$ for (1) (see Section 3). Further, we wish to analyse and compare conditions of both these criterions (see Section 5).

3. TWO TYPES OF STABILITY CONDITIONS FOR (1)

Using the notation

$$u = \frac{1+\alpha^2-\beta^2-\gamma^2}{2|\alpha-\beta\gamma|}, \quad v = \frac{1-\alpha^2+\beta^2-\gamma^2}{2|\beta-\alpha\gamma|} \tag{11}$$

we start with

Theorem 3.3. *Let α, β, γ be real scalars such that $\beta - \alpha\gamma \neq 0$ and let $k > 2$ be a positive odd integer. Then* (1) *is asymptotically stable if and only if*

$$|\beta + \gamma| < 1 + \alpha \,, \tag{12}$$

and either

$$v \geq 1 \tag{13}$$

or

$$0 < v < 1, \quad u > \cos(2\psi) \,, \tag{14}$$

where ψ is the solution in the interval $(0, \pi/(2k))$ of the equation

$$\frac{\cos((k+2)x)}{\cos(kx)} = \frac{\alpha^2 - \beta^2}{|\alpha - \beta\gamma|} \,. \tag{15}$$

Remark 3.4. The interval for the solution ψ of (15) can be more specified in some special cases. In particular, if $\alpha^2 - \beta^2 < 0$ then $\psi \in (\pi/(2k+4), \pi/(2k))$. Similarly, if $\alpha^2 - \beta^2 > 0$ then $\psi \in (0, \pi/(2k+4))$. The remaining case $\alpha^2 - \beta^2 = 0$ is trivial.

Remark 3.5. Theorem 3.3 is a direct generalization of Theorem 2.1. Indeed, if $\beta = 0$ then the system of conditions (12)–(15) is equivalent to that of Theorem 2.1. In particular, the auxiliary equation (10) can be obtained from (15) by use of $\phi = \pi/2 - \psi$ (which implies also a shift in the corresponding interval specified with respect to Remark 3.4). Contrary to the form of Theorem 2.1, we prefer here the notation utilizing symbols u and v to make this type of conditions as close as possible to the next type, which generalizes Theorem 2.2.

Theorem 3.6. *Let α, β, γ be real scalars such that $\beta - \alpha\gamma \neq 0$ and let $k > 2$ be a positive odd integer. Then* (1) *is asymptotically stable if and only if* (12) *holds, and either* (13) *or*

$$0 < v < 1 \,, \quad u > \cos\left(2\,\frac{\arcsin v}{k}\right) \,. \tag{16}$$

Remark 3.7. In the case $\beta - \alpha\gamma = 0$, (1) is asymptotically stable if and only if $|\beta + \gamma| < 1 + \alpha$ and $|\beta - \gamma| < 1 - \alpha$.

Remark 3.8. If we substitute the notation (11), then (16) can be equivalently rewritten as

$$0 < 1 - \alpha^2 + \beta^2 - \gamma^2 < 2|\beta - \alpha\gamma| \tag{17}$$

and

$$k < \frac{2\arcsin\big((1 - \alpha^2 + \beta^2 - \gamma^2)/(2|\beta - \alpha\gamma|)\big)}{\arccos\big((1 + \alpha^2 - \beta^2 - \gamma^2)/(2|\alpha - \beta\gamma|)\big)} \,. \tag{18}$$

Notice that the latter inequality provides with an explicit stability condition on the delay k. This condition also determines the set $\widetilde{S}(k)$ consisting of all real triplets α, β, γ such that the asymptotic stability property of (1) is actually depending on odd k. It holds

$$\widetilde{S}(k) = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : (12),\ (17) \text{ and } (18) \text{ hold}\} \,.$$

Similarly, under the assumptions of Theorem 3.6, the part S_{odd} of the asymptotic stability region for (1), which is independent of any k odd, is given by (12) and (13), i.e.

$$S_{odd} = \{(\alpha,\beta,\gamma) \in \mathbb{R}^3 : \quad |\beta+\gamma| < 1+\alpha, \quad |\beta-\gamma| \le 1-\alpha\}\,.$$

Of course, $S(k) = S_{odd} \cup \widetilde{S}(k)$ for k odd. A more detailed analysis shows that if we restrict only on triplets α, β, γ with $\alpha < 0$ and $\beta\gamma > 0$, then $C = S_{odd} = S(k)$, i.e. the generalized Clark's condition (8) is optimal in this case.

Remark 3.9. As we have already noted in the introductory section, the case k even is covered (after some simple rearrangements) by the relevant results of [**2**]. To complete the description of stability sets, we present now the following reformulations of Theorem 1.2 and Theorem 1.3 of [**2**] for the case of (1).

First let k be even and $k/2$ be odd. Then (1) is asymptotically stable if and only if $|\alpha+\beta| < 1+\gamma$, and either

$$\gamma - 1 < |\alpha-\beta| \le 1-\gamma$$

or

$$|\alpha-\beta| > |1-\gamma|, \qquad k < \frac{2\arccos\big((-1+\alpha^2-\beta^2+\gamma^2)/(2|\alpha\gamma-\beta|)\big)}{\arccos\left((1+\alpha^2-\beta^2-\gamma^2)/(2|\alpha-\beta\gamma|)\right)}\,.$$

Now let k as well as $k/2$ be even. Then (1) is asymptotically stable if and only if $|\alpha+\gamma| < 1+\beta$, and either

$$\beta - 1 < |\alpha-\gamma| \le 1-\beta$$

or

$$|\alpha-\gamma| > |1-\beta|, \qquad k < \frac{2\arccos\big((-1+\alpha^2-\beta^2+\gamma^2)/(2|\alpha\gamma-\beta|)\big)}{\arccos\left((1+\alpha^2-\beta^2-\gamma^2)/(2|\alpha-\beta\gamma|)\right)}\,.$$

Hence, analogously to the case k odd, we get

$$S_{even} = \{(\alpha,\beta,\gamma) \in \mathbb{R}^3 : \quad |\alpha+\beta| < 1+\gamma, \quad \gamma-1 < \beta-\alpha \le 1-\gamma\}\,.$$

Summarizing both the parity cases, we can describe the delay-independent stability region for (1) as

$$S = S_{odd} \cap S_{even} = \{(\alpha,\beta,\gamma) \in \mathbb{R}^3 : \quad |\beta+\gamma| < 1+\alpha, \quad |\beta-\gamma| < 1-\alpha\}\,.$$

4. PROOFS

A direct proof of Theorem 3.3 is a pretty long and tedious matter. Instead, we start with the proof of Theorem 3.6 and then we verify the conditions of Theorem 3.3 by use of that assertion.

Proof of Theorem 3.6. We employ here the Schur-Cohn test and analyse its conditions guaranteeing that all the zeros of (2) are located inside the unit circle.

This procedure requires to introduce the following special matrices. For any $r = 0, 1, \dots, k+1$, we denote by $F_{q,r}$ $(q = 0, 1, \dots, r-1)$ $r \times r$ matrices with the unit elements on the $(q+j)$-th row and j-th column position $(j = 1, 2, \dots, r-q)$, and with the zero elements elsewhere. Similarly, we denote by $G_{q,r}$ $(q = 0, 1, \dots, r-1)$ $r \times r$ matrices with the unit elements on the $(q+j)$-th row and $(r-j+1)$-st column position ($j = 1, 2, \dots, r-q$), and with the zero elements elsewhere. Utilizing this we can define $r \times r$ matrices

$$B_r^{\pm} = F_{0,r} + \alpha F_{2,r} \pm \gamma G_{0,r} \pm \beta G_{2,r}$$

and

$$\widehat{B}_{k+1}^{+} = B_{k+1}^{+} + \beta F_{k,k+1} + \alpha G_{k,k+1}\,,$$
$$\widehat{B}_{k+1}^{-} = B_{k+1}^{-} + \beta F_{k,k+1} - \alpha G_{k,k+1}\,.$$

Now the Schur-Cohn criterion can be reformulated for $p(\lambda)$ with k odd as follows (for its general version we refer to [**7**] or [**15**]).

Proposition 4.10. *Let $k > 2$ be odd. Then all the zeros of* (2) *lie inside the unit circle if and only if the following conditions are satisfied simultaneously.*

(a) $|\beta + \gamma| < 1 + \alpha$;

(b) $\det(B_{2\ell}^{+}) > 0, \quad \det(B_{2\ell}^{-}) > 0, \qquad \ell = 1, 2, \dots, (k-1)/2$;

(c) $\det(\widehat{B}_{k+1}^{+}) > 0, \quad \det(\widehat{B}_{k+1}^{-}) > 0\,.$

We illustrate the structure of matrices involved in Proposition 4.10 (and thus also an efficiency of this criterion) via the choice $k = 5$.

Example 4.11. We consider the linear difference equation

$$y(n+2) + \alpha\, y(n) + \beta\, y(n-3) + \gamma\, y(n-5) = 0, \qquad n = 0, 1, 2, \dots \tag{19}$$

with the characteristic polynomial

$$p(\lambda) \equiv \lambda^7 + \alpha\,\lambda^5 + \beta\,\lambda^2 + \gamma\,. \tag{20}$$

To analyse its zeros location, we utilize the matrices $B_2^{\pm}$, $B_4^{\pm}$ and $\widehat{B}_6^{\pm}$ in the form

$$B_2^{\pm} = \begin{pmatrix} 1 & \pm\gamma \\ \pm\gamma & 1 \end{pmatrix}, \qquad B_4^{\pm} = \begin{pmatrix} 1 & 0 & 0 & \pm\gamma \\ 0 & 1 & \pm\gamma & 0 \\ \alpha & \pm\gamma & 1 & \pm\beta \\ \pm\gamma & \alpha & \pm\beta & 1 \end{pmatrix}$$

and

$$\widehat{B}_6^{\pm} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \pm\gamma \\ 0 & 1 & 0 & 0 & \pm\gamma & 0 \\ \alpha & 0 & 1 & \pm\gamma & 0 & \pm\beta \\ 0 & \alpha & \pm\gamma & 1 & \pm\beta & 0 \\ 0 & \pm\gamma & \alpha & \pm\beta & 1 & 0 \\ \beta \pm \gamma & 0 & \pm\beta & \alpha & 0 & 1 \pm \alpha \end{pmatrix}.$$

By Proposition 4.10, all the zeros of (20) lie inside the unit circle if and only if $|\beta+\gamma| < 1+\alpha$ and

$$\det(B_2^{+}) = \det(B_2^{-}) = 1 - \gamma^2 > 0\,,$$
$$\det(B_4^{+}) = \det(B_4^{-}) = (1 + \alpha^2 - \beta^2 - \gamma^2)(1 - \gamma^2) - (\alpha - \beta\gamma)^2 > 0\,,$$
$$\det(\widehat{B}_6^{+}) > 0\,, \qquad \det(\widehat{B}_6^{-}) > 0$$

(we omit here explicit expansions of $\det(\widehat{B}_6^+)$ and $\det(\widehat{B}_6^-)$ in terms of α, β and γ due to their lengthy computational forms). Equivalently, these conditions are necessary and sufficient for asymptotic stability of (19).

This example well demonstrates that explicit reformulations of the conditions of Proposition 4.10 become very complicated with increasing values of k. Therefore, it would be useful to rewrite conditions (b) and (c) of Proposition 4.10 into a more efficient form.

To illustrate the main idea of the procedures that follow, we consider the matrices $B_2^{\pm}$, $B_4^{\pm}$ described and utilized in Example 4.11. It is easy to check that

$$\det(B_4^{\pm}) = (1+\alpha^2-\beta^2-\gamma^2)\det(B_2^{\pm}) - (\alpha-\beta\gamma)^2\,,$$

which can be formally rewritten as

$$\det(B_4^{\pm}) - (1+\alpha^2-\beta^2-\gamma^2)\det(B_2^{\pm}) + (\alpha-\beta\gamma)^2\det(B_0^{\pm}) = 0\,,$$

where $B_0^{\pm}$ are empty matrices with $\det(B_0^{\pm}) = 1$. In the sequel, we show that this relation is fundamental in our effort to make Proposition 4.10 more applicable. In particular, it forms a base for a special recurrence between the determinants $B_{2\ell}^{\pm}$, $B_{2\ell+2}^{\pm}$ and $B_{2\ell+4}^{\pm}$ with a general $\ell = 0,1,\dots,(k-3)/2$. Furthermore, using the related procedures we are able to express $\det(\widehat{B}_{k+1}^+)$ and $\det(\widehat{B}_{k+1}^-)$ in terms of $\det(B_{k+1}^{\pm})$ and $\det(B_{k-1}^{\pm})$.

The precision of previous considerations enables us to formulate the following assertion, which is very useful in our next analysis.

Proposition 4.12. *The conditions* **(b)**, **(c)** *of Proposition* 4.10 *are equivalent to*:

(d) *The solution $z(\ell)$ of the initial value problem*

$$z(\ell+2) - (1+\alpha^2-\beta^2-\gamma^2)z(\ell+1) + (\alpha-\beta\gamma)^2 z(\ell) = 0\,, \tag{21}$$

$$z(0)=1, \qquad z(1) = 1-\gamma^2 \tag{22}$$

satisfies

$$z(\ell) > 0 \qquad \text{for all } \ell = 1,2,\dots,(k-1)/2 \tag{23}$$

and

$$z((k+1)/2) - |\alpha-\beta\gamma|\,z((k-1)/2) > 0\,. \tag{24}$$

Proof. We show that for nonnegative integers ℓ it holds that

$$\det(B_{2\ell+4}^{\pm}) - (1+\alpha^2-\beta^2-\gamma^2)\det(B_{2\ell+2}^{\pm}) + (\alpha-\beta\gamma)^2\det(B_{2\ell}^{\pm}) = 0\,. \tag{25}$$

Indeed, the addition of the $(-\alpha)$ multiple of the first row to the third one, and then the addition of the $(\mp\beta)$ multiple of the third row to the last one yield

$$\det(B^{\pm}_{2\ell+4}) = \det\left(\begin{array}{c|ccc|c} 1 & & \mathbf{o} & & \pm\gamma \\ \hline & & & & 0 \\ & & & & \pm\beta \mp \alpha\gamma \\ \mathbf{o}^T & & B^{\pm}_{2\ell+2} & & \\ & & & & \mathbf{o}^T \\ \hline \pm\gamma & \mathbf{o} & \alpha-\beta\gamma & 0 & 1-\beta^2+\alpha\beta\gamma \end{array}\right),$$

where $\mathbf{o}$ and $\mathbf{o}^T$ are row and column zero vectors of appropriate dimensions. The Laplace expansions along the first row and then along the first column give

$$\det(B^{\pm}_{2\ell+4}) = -\gamma^2 \det(B^{\pm}_{2\ell+2}) + \det\left(\begin{array}{ccc|c} & & & 0 \\ & & & \pm\beta \mp \alpha\gamma \\ & B^{\pm}_{2\ell+2} & & \\ & & & \mathbf{o}^T \\ \hline \mathbf{o} & \alpha-\beta\gamma & 0 & 1-\beta^2+\alpha\beta\gamma \end{array}\right).$$

Now we rearrange the matrix from the previous equation by adding the α multiple of the $(2\ell+1)$-st column and $(\mp\beta)$ multiple of the second column to the last but one column. Using the Laplace expansions along the last but one column of this rearranged matrix, then along the last row of the obtained minor and finally along the second row of the other obtained minor we obtain

$$\begin{aligned} \det(B^{\pm}_{2\ell+4}) = \;& (1+\alpha^2-\beta^2-\gamma^2)\det(B^{\pm}_{2\ell+2}) \\ & -(\alpha-\beta\gamma)^2 \det\left(\begin{array}{c|ccc|c} 1 & & \mathbf{o} & & \pm\gamma \\ \hline \alpha & & & & \pm\beta \\ \mathbf{o}^T & & B^{\pm}_{2\ell-2} & & \mathbf{o}^T \\ & & & & \\ \hline \pm\gamma & \pm\beta & \mathbf{o} & \alpha & 1 \end{array}\right). \end{aligned}$$

Notice that the matrix from the previous equation and the matrix $B^{\pm}_{2\ell}$ have the same value of determinant, which can be verified by use of an even number of appropriate row and column interchanges. Consequently, (25) holds.

The previous procedure is formally applicable only if $\ell = 4, 5, \dots$. For the remaining values of nonnegative integers ℓ ($\ell = 0, 1, 2, 3$), the validity of (25) can be easily verified by a direct computation, where the term $B^{\pm}_0$ means the empty matrix with $\det(B^{\pm}_0) = 1$ (see the considerations preceding Proposition 4.12).

Thus the relation (25) with the starting values

$$\det(B^{\pm}_0) = 1, \qquad \det(B^{\pm}_2) = 1 - \gamma^2 \tag{26}$$

yields the recurrence, which enables us to obtain the values of $\det(B^{+}_{2\ell})$ and $\det(B^{-}_{2\ell})$ for all $\ell = 0, 1, \dots, (k+1)/2$. Obviously, $\det(B^{+}_{2\ell}) = \det(B^{-}_{2\ell})$, because the determinants of $B^{+}_{2\ell}$ and $B^{-}_{2\ell}$ ($\ell = 0, 1, \dots, (k+1)/2$) satisfy (25) with the same conditions

(26). If we introduce the notation $z(\ell) = \det(B^{\pm}_{2\ell})$ for $\ell = 0, 1, 2, \ldots$, then (25) and (26) represent an initial value problem for the difference equation (21) subject to the initial conditions (22).

It remains to analyse the positivity of $\det(\widehat{B}^{+}_{k+1})$ and $\det(\widehat{B}^{-}_{k+1})$ in terms of the solution of (21), (22). Since $\widehat{B}^{\pm}_{k+1}$ can be obtained by addition of the vector $(\beta, \mathbf{o}, \pm\alpha)$ to the last row of $B^{\pm}_{k+1}$, then

$$\det(\widehat{B}^{\pm}_{k+1}) = \det(B^{\pm}_{k+1}) \pm (\alpha - \beta\gamma) \det(B^{\pm}_{k-1}) = z\left((k+1)/2\right) \pm (\alpha - \beta\gamma) z\left((k-1)/2\right) .$$

Hence, if $\det(B^{\pm}_{k-1}) > 0$, then the simultaneous positivity of both $\det(\widehat{B}^{+}_{k+1})$ and $\det(\widehat{B}^{-}_{k+1})$ is equivalent to (24). □

To simplify the next procedures, we introduce the notation

$$R_1 = 1+\alpha+\beta+\gamma, \quad R_2 = 1+\alpha-\beta-\gamma, \quad R_3 = 1-\alpha+\beta-\gamma, \quad R_4 = 1-\alpha-\beta+\gamma$$

and

$$L = 1 + \alpha^2 - \beta^2 - \gamma^2 = (R_1R_2 + R_3R_4)/2\,, \tag{27}$$

$$M = 1 - \alpha^2 + \beta^2 - \gamma^2 = (R_1R_3 + R_2R_4)/2\,, \tag{28}$$

$$D = L^2 - 4(\alpha - \beta\gamma)^2 = M^2 - 4(\beta - \alpha\gamma)^2 = R_1R_2R_3R_4\,. \tag{29}$$

By Proposition 4.10 and Proposition 4.12, it is enough to show that the properties **(a)** and **(d)** are equivalent to the conditions of Theorem 3.6. Since **(a)** and (12) are identical inequalities, we assume the validity of **(a)** (equivalently $R_1 > 0$ and $R_2 > 0$) and analyse the condition **(d)** with respect to the following three sign cases.

Case 1. Let $R_3R_4 > 0$. Then $D > 0$ and the solution of the initial value problem (21), (22) becomes

$$z(\ell) = c_1 \left(\frac{1}{2}(L + D^{1/2})\right)^{\ell} + c_2 \left(\frac{1}{2}(L - D^{1/2})\right)^{\ell}, \qquad \ell = 0, 1, \ldots ,$$

where $c_1 = (D^{1/2} + M)/(2D^{1/2})$ and $c_2 = (D^{1/2} - M)/(2D^{1/2})$. If $\alpha - \beta\gamma \neq 0$ then the condition (24) becomes

$$c_1 \left(\frac{L + D^{1/2}}{L - D^{1/2}}\right)^{(k-1)/2} > c_2 \frac{D^{1/2} - L + 2|\alpha - \beta\gamma|}{D^{1/2} + L - 2|\alpha - \beta\gamma|}\,. \tag{30}$$

Employing (27) and (29) it is easy to show that nominators and denominators of both the fractions in (30) are positive.

If $R_3 < 0$ and $R_4 < 0$, i.e. if $v < -1$, then $M + D^{1/2} < 0$ due to (29). Hence, $c_1 < 0 < c_2$ and (30) cannot be satisfied for any positive integer k.

Contrary, if $R_3 > 0$ and $R_4 > 0$, i.e. if $v > 1$, then $M - D^{1/2} > 0$. Consequently, $c_1 > 0 > c_2$ and (30) is satisfied trivially for any positive k. This fact along with $z(0) = 1$ implies that (23) and (24) hold for any positive integer k.

The case $\alpha - \beta\gamma = 0$ can be discussed analogously with the same conclusion.

Case 2. Let $R_3R_4 = 0$, i.e. let $v = \pm 1$ (the case $v = 0$ is excluded due to $\beta - \alpha\gamma \neq 0$). Then $D = 0$ and the solution of (21), (22) is

$$z(\ell) = \left(1 + \frac{M}{L}\ell\right)\left(\frac{L}{2}\right)^{\ell}, \qquad \ell = 0, 1, \dots .$$

Moreover, $\alpha - \beta\gamma > 0$ due to $L/2 - (\alpha - \beta\gamma) = R_3R_4/2 = 0$ and $L = R_1R_2/2 > 0$. Substituting $z(\ell)$ into (24) one gets

$$\left(1 + \frac{M(k+1)}{2L}\right)\frac{L}{2} > (\alpha - \beta\gamma)\left(1 + \frac{M(k-1)}{2L}\right).$$

Equivalently,

$$\left(\frac{M}{2} - \frac{M}{L}(\alpha - \beta\gamma)\right)\frac{k+1}{2} > -\frac{M}{L}(\alpha - \beta\gamma)\,,$$

i.e.

$$R_3R_4M(k+1) > -4M(\alpha - \beta\gamma)\,.$$

From here we get that (24) holds if and only if $M > 0$. Taking into account (28) and the assumption $R_3R_4 = 0$, (24) holds for any positive integer k if and only if $v = 1$ (i.e. if and only if either $R_3 > 0$, $R_4 = 0$ or $R_3 = 0$, $R_4 > 0$) . This also implies the validity of (23) for any positive integer k.

Summarizing Case 1 and Case 2, we obtain the condition (13).

Case 3. Let $R_3R_4 < 0$, i.e. let $-1 < v < 1$. Then $D < 0$ and $\alpha - \beta\gamma > 0$ due to $\alpha - \beta\gamma = (R_1R_2 - R_3R_4)/4$. The solution $z(\ell)$ of (21), (22) becomes

$$z(\ell) = \frac{2|\beta - \alpha\gamma|}{(-D)^{1/2}}(\alpha - \beta\gamma)^{\ell}\cos(\varphi + \vartheta\ell), \qquad \ell = 0, 1, \dots , \tag{31}$$

where

$$\varphi = \operatorname{arctan}\frac{-M}{(-D)^{1/2}}, \qquad \vartheta = \operatorname{arccot}\frac{L}{(-D)^{1/2}}\,.$$

It is easy to verify that $z(\ell) > 0$ for $\ell = 0, 1, \dots, s$ and $z(s+1) \leq 0$ if $s = \lceil(\pi/2 - \varphi)/\vartheta\rceil - 1$, where $\lceil\cdot\rceil$ is the ceiling function. Therefore we analyse (24) for $k = 3, 5, \dots, 2s + 1$.

If we substitute (31) into (24), then

$$\cos(\varphi + \vartheta(k+1)/2) > \cos(\varphi + \vartheta(k-1)/2)\,,$$

which can be written as

$$\cot(\vartheta(k+1)/2)\sin(\vartheta/2 - \varphi) > \cos(\vartheta/2 - \varphi) \tag{32}$$

due to $\sin(\vartheta/2) > 0$. Because of the considered domain of k, φ and ϑ, this relation cannot be satisfied in the case of $\vartheta/2 - \varphi \le 0$. Thus from (32) we obtain the upper bound on k in the form

$$k < -\frac{2\varphi}{\vartheta} = \frac{2\arctan(M/(-D)^{1/2})}{\operatorname{arccot}(L/(-D)^{1/2})}\,. \tag{33}$$

We emphasize that the righthand side of (33) is positive if and only if $M > 0$ (i.e. if and only if $v > 0$). Moreover, any k odd satisfying (33) cannot exceed $2s+1$. Consequently, (33) serves as an active bound on k. To summarize, the conditions on v imply $(16)_1$, and the relation (33) along with standard formulae for cyclometric functions and the notation (11) imply $(16)_2$. This proves Theorem 3.6.

Using the same argumentation we can discuss the case $\beta - \alpha\gamma = 0$ (see Remark 3.7). In such a case, it is enough to consider only the condition $R_3 > 0$ and $R_4 > 0$ (analysis of the remaining sign variants leads to a contradiction).

Proof of Theorem 3.3. It is enough to show the equivalence of (14) and (16). We put $\omega = (\arccos u)/2$ and first check that this symbol is well-defined. Indeed, if we rewrite u, v as

$$u = \frac{R_1R_2 + R_3R_4}{|R_1R_2 - R_3R_4|}, \quad v = \frac{R_1R_3 + R_2R_4}{|R_1R_3 - R_2R_4|}\,,$$

then $-1 < u < 1$ by virtue of $0 < v < 1$. Further, we define the function

$$f(x) = \frac{\cos((k+2)x)}{\cos(kx)} - \frac{\alpha^2 - \beta^2}{|\alpha - \beta\gamma|}, \quad x \in \left(0, \frac{\pi}{2k}\right)\,.$$

This function is continuous and decreasing in $(0, \pi/(2k))$, and has a (unique) zero in this interval due to

$$f(0) = 1 - \frac{\alpha^2 - \beta^2}{|\alpha - \beta\gamma|} > 1 - u > 0, \quad \text{and} \quad \lim_{x \to \frac{\pi}{2k}^-} f(x) = -\infty\,.$$

Moreover,

$$\operatorname{sgn} f(\pi/(2k+4)) = \operatorname{sgn}(\beta^2 - \alpha^2)\,,$$

which implies specifications of this interval stated in Remark 3.4. Using this we can reformulate (14) as

$$0 < v < 1, \quad \cos(2\omega) > \cos(2\psi)\,,$$

where $\psi \in (0, \pi/(2k))$ is such that $f(\psi) = 0$. Equivalently,

$$0 < v < 1, \quad f(\omega) > 0\,. \tag{34}$$

Using appropriate formulae for goniometric and cyclometric functions we get

$$\begin{aligned} f(\omega) &= u - \tan(k\omega)\sin(2\omega) - \frac{\alpha^2-\beta^2}{|\alpha-\beta\gamma|} \\ &= -\tan(k\omega)\sin(2\omega) + \frac{v}{(1-v^2)^{1/2}}(1-u^2)^{1/2} \\ &= -\tan(k\omega)\sin(2\omega) + \frac{v}{(1-v^2)^{1/2}}\sin(2\omega) \\ &= \sin(2\omega)\left(-\tan\left(\frac{k}{2}\arccos(u)\right) + \tan(\arcsin v)\right) . \end{aligned}$$

Since $\sin(2\omega) > 0$, substituting this into (34) we get (16). Theorem 3.3 is proved.

5. SOME CONSEQUENCES, EXAMPLES AND COMPARISONS

In this section, we compare two types of stability conditions formulated in Section 3 - both from the analytical and geometrical viewpoint. We recall that these conditions (see Theorem 3.3 and Theorem 3.6) have been formulated with the intention to make this comparison as clear as possible.

We start with some examples illustrating that the system of conditions involved in Theorem 3.6 is more convenient than that of Theorem 3.3. Of course, the main problem connected with the conditions (12)–(15) of Theorem 3.3 is connected with the nonlinear equation (15). The only case, when this equation allows an explicit form of the solution, is $|\alpha| = |\beta|$. Then $\psi = \pi/(2k+4)$ and Theorem 3.3 can yield a fully explicit stability criterion. We formulate it for the case $\alpha = \beta$ (the case $\alpha = -\beta$ is analogous).

Corollary 5.13. *Let α, γ be real constants and $k > 2$ be a positive odd integer. Then*

$$y(n+2) + \alpha\,(y(n) + y(n-k+2)) + \gamma\, y(n-k) = 0, \qquad n = 0, 1, 2, \dots$$

is asymptotically stable if and only if

$$-2\alpha - 1 < \gamma < 1 \tag{35}$$

and either

$$\gamma \geq 2|\alpha| - 1 \tag{36}$$

or

$$2|\alpha|\cos\frac{\pi}{k+2} - 1 < \gamma < 2|\alpha| - 1\,. \tag{37}$$

Proof. If $\alpha = \beta$ then (11) becomes $u = v = (1+\gamma)/2|\alpha|$ with respect to (12). Since $\psi = \pi/(2k+4)$, the conditions (12)–(14) can be trivially simplified to (35)–(37), respectively. □

If $|\alpha| \neq |\beta|$ then (15) has to be solved numerically. In this case, the conditions of Theorem 3.3 enable easily to check the asymptotic stability property of (1) with any fixed entries α, β, γ and k. Moreover, in a particular case when $\beta = 0$ and α, k are fixed, Theorem 3.3 allows to establish the stability interval for the remaining free parameter γ.

EXAMPLE 5.14. We investigate the asymptotic stability property of

$$y(n+2) + \frac{1}{2}\, y(n) + \gamma\, y(n-5) = 0, \qquad n = 0, 1, 2, \dots \tag{38}$$

with respect to the value of γ. In this case, (15) becomes

$$\frac{\cos(7x)}{\cos(5x)} = \frac{1}{2}$$

with the unique zero $\psi \approx 0.1789$ lying in $(0, \pi/14)$ (see Remark 3.4). Since $u = 5/4 - \gamma^2$ and $v = (3/4 - \gamma^2)/|\gamma|$, the conditions (12)–(15) can be reduced to

$$|\gamma| < 3/2, \qquad \gamma^2 < 5/4 - \cos(0.3579)\,.$$

Then, by Theorem 3.3, (38) is asymptotically stable if and only if

$$\gamma \in (-0.5598, 0.5598)\,. \tag{39}$$

REMARK 5.15. The difference equation (38) is a particular case of (19), whose stability properties have been investigated using the Schur-Cohn test (see Proposition 4.10) in Example 4.11. Stability analysis of (38) via this test implies the conditions $|\gamma| < 3/2$ and

$$\begin{aligned}
\det(B_2^+) &= \det(B_2^-) = 1 - \gamma^2 > 0\,,\\
\det(B_4^+) &= \det(B_4^-) = 1 - (9/4)\gamma^2 + \gamma^4 > 0\,,\\
\det(\widehat{B}_6^+) &= 3/2 - (75/16)\gamma^2 + 4\gamma^4 - \gamma^6 > 0\,,\\
\det(\widehat{B}_6^-) &= 1/2 - (39/16)\gamma^2 + 3\gamma^4 - \gamma^6 > 0\,,
\end{aligned}$$

which yield, after some necessary calculations, the stability interval (39). It is obvious that if we consider the same problem with higher orders k, then calculations related to Proposition 4.10 become very complicated (with increasing k), while the computational difficulty of procedures related to Theorem 3.3 is independent of k.

Now we turn our attention to the applicability of Theorem 3.6. First we show how its system of conditions imply the result of Example 5.14. Doing this, we employ the conditions of Theorem 2.2, which is a special case of Theorem 3.6 covering the choice $\beta = 0$. By these conditions, (38) is asymptotically stable if and only if $|\gamma| \leq 1/2$ or

$$1/2 < |\gamma| < \sqrt{1/2}, \quad 5/2 < \arcsin \frac{3/4 - \gamma^2}{|\gamma|} \Big/ \arccos(5/4 - \gamma^2)\,.$$

Solving the last inequality as the (nonlinear) equation with respect to $|\gamma|$, we get $|\gamma| \approx 0.5598$. This leads to the stability interval (39).

Contrary to Theorem 3.3, one can utilize the system of conditions of Theorem 3.6 also in other cases when two of entry parameters α, β, γ are fixed (along with the value of k) and the stability interval for the remaining free parameter is to be found. However, the main contribution of Theorem 3.6 consists in a different aspect as the following example illustrates.

Example 5.16. We consider the linear difference equation

$$y(n+2)+\frac{1}{2}\,y(n)+\frac{1}{3}\,y(n-k+2)-\frac{3}{16}\,y(n-k)=0, \qquad n=0,1,2,\dots, \tag{40}$$

whose coefficients α, β, γ are fixed, but the value of a positive integer $k>2$ is not specified. Then $v\approx 0.9670\in(0,1)$, hence the asymptotic stability condition imposed on integer odd k can be expressed via (18) as $k<13.4845$. Further, if both k and $k/2$ are even, then $k<29.6158$ due to Remark 3.9. On the other hand, if k is even and $k/2$ odd, then the condition $|\alpha+\beta|<1+\gamma$ is not satisfied, i.e. (40) is not asymptotically stable for any k even such that $k/2$ is odd.

To summarize, Theorem 3.6 (supplied with Remark 3.8) and Remark 3.9 imply that (40) is asymptotically stable if and only if

$$k\in\{3,4,5,7,8,9,11,12,13,16,20,24,28\}\,.$$

Now we support these illustrating examples by some general observations. Analytically, the only distinction between both the system of conditions of Theorem 3.3 and Theorem 3.6 consists in inequalities $(14)_2$ and $(16)_2$. While the latter condition is fully explicit in terms of α, β, γ and k (see also Remark 3.8), the condition $(14)_2$ requires a solution of the auxiliary nonlinear equation (15). This makes the system presented in Theorem 3.6 much more convenient. In particular, if we are given with fixed values of α, β, γ and look for a critical value of k when (1) definitely loses its asymptotic stability property, then (18) explicitly provides with such a value, while the condition $(14)_2$ does not answer this problem.

Note also that the condition (16) can be formally rewritten as

$$0<v<1, \quad u>\cos(2\psi)\,,$$

where ψ is the solution in the interval $(0,\pi/(2k))$ of the equation

$$\sin(kx)=v\,. \tag{41}$$

Thus the systems of conditions of Theorem 3.3 and Theorem 3.6 are completely unified with an obvious convenience of (41) compared to (15). We emphasize that $\psi=(\arcsin v)/k$ is not the solution of (15), except for the case, when $(16)_2$ is satisfied in the form of the equality, i.e. when $u=\cos(2(\arcsin v)/k)$.

A comparison from the geometrical viewpoint is more interesting. To simplify it, we first put $\alpha=0$ in (1) and consider the difference equation of advanced type

$$y(n+2)+\beta\,y(n-k+2)+\gamma\,y(n-k)=0, \qquad n=0,1,2,\dots. \tag{42}$$

A direct application of Theorem 3.3 (along with Remark 3.4) yields

Corollary 5.17. *Let β, γ be real scalars and $k > 2$ be a positive odd integer. Then* (42) *is asymptotically stable if and only if*

$$|\beta + \gamma| < 1\,, \tag{43}$$

and either

$$|\beta - \gamma| \le 1 \tag{44}$$

or

$$|\beta - \gamma| > 1, \quad 1 - \beta^2 - \gamma^2 > 2|\beta\gamma|\cos(2\psi)\,, \tag{45}$$

where ψ is the solution in the interval $(\pi/(2k+4), \pi/(2k))$ of the equation

$$\frac{\cos((k+2)x)}{\cos(kx)} = -\frac{|\beta|}{|\gamma|}\,.$$

Similarly, Theorem 3.6 implies

Corollary 5.18. *Let β, γ be real scalars and $k > 2$ be a positive odd integer. Then* (42) *is asymptotically stable if and only if* (43), *and either* (44) *or*

$$|\beta - \gamma| > 1, \quad 2\arcsin\frac{1 + \beta^2 - \gamma^2}{2|\beta|} - k\arccos\frac{1 - \beta^2 - \gamma^2}{2|\beta\gamma|} > 0$$

hold.

Using these assertions we can describe the corresponding (delay-dependent) stability region $S(k)$ for (42) as a part of the (β, γ)-plane. A more detailed analysis reveals that the boundary of $S(k)$ consists of two lines

$$\ell_1 : \beta + \gamma + 1 = 0, \qquad \ell_2 : \beta + \gamma - 1 = 0$$

and two curves η_1, η_2. By Corollary 5.17, we obtain their analytical expressions via the solution of the system

$$\frac{1 - \beta^2 - \gamma^2}{2|\beta\gamma|} = \cos(2\psi), \quad \frac{\cos((k+2)\psi)}{\cos(k\psi)} = -\frac{|\beta|}{|\gamma|}, \quad \psi \in \left(\frac{\pi}{2k+4}, \frac{\pi}{2k}\right)$$

with unknowns $|\beta|$ and $|\gamma|$ (ψ is now taken for a parameter). Considering (43) and $(45)_1$, these boundary curves are given by

$$\eta_i : \beta = \pm\frac{\cos((k+2)\psi)}{\sin(2\psi)}, \quad \gamma = \pm\frac{\cos(k\psi)}{\sin(2\psi)}, \qquad \psi \in \left(\frac{\pi}{2k+4}, \frac{\pi}{2k}\right)\,,$$

$i = 1, 2$. On the other hand, Corollary 5.18 describes η_1, η_2 as parts of the curve $F(\beta, \gamma) = 0$, where

$$F(\beta, \gamma) = 2\arcsin\frac{1 + \beta^2 - \gamma^2}{2|\beta|} - k\arccos\frac{1 - \beta^2 - \gamma^2}{2|\beta\gamma|}\,,$$

restricted with respect to (43) and $(45)_1$. To summarize it, Corollary 5.17 can establish the parametrical form of these boundary curves, while Corollary 5.18 the implicit one.

A similar geometrical interpretation can be performed also when $\alpha \neq 0$, i.e. for the four-term equation (1). Then the asymptotic stability set $S(k)$ is a part of the (α, β, γ)-space bounded by the planes

$$p_1 : \alpha + \beta + \gamma + 1 = 0, \qquad p_2 : \alpha - \beta - \gamma + 1 = 0$$

and by surfaces σ_1, σ_2. While the analytical description of these surfaces based on Theorem 3.3 can be read as

$$G(\alpha, \beta, \gamma, \psi) = 0 \quad \textit{and} \quad H(\alpha, \beta, \gamma, \psi) = 0, \qquad \psi \in \left(0, \frac{\pi}{2k}\right)$$

with formulae for G and H given by

$$G(\alpha, \beta, \gamma, \psi) = \frac{1 + \alpha^2 - \beta^2 - \gamma^2}{2|\alpha - \beta\gamma|} - \cos(2\psi),$$
$$H(\alpha, \beta, \gamma, \psi) = \frac{\alpha^2 - \beta^2}{|\alpha - \beta\gamma|} - \frac{\cos((k+2)\psi)}{\cos(k\psi)}\,,$$

Theorem 3.6 yields this description in the implicit form $F(\alpha, \beta, \gamma) = 0$, where

$$F(\alpha, \beta, \gamma) = 2 \arcsin \frac{1 - \alpha^2 + \beta^2 - \gamma^2}{2|\beta - \alpha\gamma|} - k \arccos \frac{1 + \alpha^2 - \beta^2 - \gamma^2}{2|\alpha - \beta\gamma|}\,.$$

6. CONCLUDING REMARKS

The main goal of this paper was twofold. First, the formulation of two asymptotic stability criterions for the delay difference equation (1). Second, discussions of these two criterions and their comparisons.

The first type of stability conditions (presented in Theorem 3.3) is prevailing in the current literature on stability of delay difference equations. In addition to Kuruklis' pioneering paper [**13**], where this type of conditions appeared for the first time, and papers [**6**], [**18**], which have been already mentioned, we can refer to papers by Dannan [**5**] and Matsunaga and Hajiri [**16**]. These papers consider various types of delay difference equations and formulate necessary and sufficient stability conditions, which require a solution of some auxiliary nonlinear equations. The form of such conditions is very close to the above cited Ren's result (Theorem 2.1). In a larger sense, we can assign here also the papers by Papanicolaou [**17**] and Kipnis and Nigmatullin [**11**], where the stability boundary of a three-term difference equation was described by use of certain parametrical curves.

On the other hand, explicit descriptions of delay-dependent stability regions, representing the second type of stability conditions, have appeared only recently in [**2**] and [**3**] in the frame of stability investigations of (5) and (9), respectively.

Advantages of this type of stability conditions have been discussed in the previous section. Therefore, it might be useful to reformulate the existing results of papers [**5**], [**16**] into such explicit conditions and, of course, to discuss these conditions for other forms of delay difference equations, which have not been considered yet. As an example may serve the equation

$$y(n+m) + \alpha\, y(n) + \beta\, y(n-k+m) + \gamma\, y(n-k) = 0, \qquad n = 0, 1, 2, \ldots$$

with positive integers $k > m$, which generalizes not only equations considered in this paper, but all other types of scalar delay difference equations studied so far with respect to necessary and sufficient stability conditions. The investigation of these problems requires, among others, further developments of the proof method proposed in [**2**] and utilized, along with some other additional procedures, also in this paper.

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