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ON A FIXED POINT THEOREM AND ITS APPLICATION IN DYNAMIC PROGRAMMING

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We present some fixed point theorems for contractions of rational type. These theorems generalize some other results appearing in the literature. Moreover, we present some examples illustrating our results. Finally, we present an application to the study of the existence and uniqueness of solutions to a class of functional equations arising in dynamic programming.

1. INTRODUCTION

In recent times, Fixed Point Theory has become one of the most useful branches of Nonlinear Analysis, mainly due to its possible applications in several areas. For instance, different classes of matrix, differential and integral equations can be solved using the appropriate techniques in this field of knowledge.

Although it was not the first result in this sense, the Banach contractive mapping principle is undoubtedly the most widely recognized theorem in this area. This result guarantees that any contractive mapping from a complete metric space into itself has a unique fixed point. After the appearance of this result in Banach's thesis in 1922, a great number of extensions have been introduced to extend and improve it in different abstract metric spaces, even endowed with a partial order.

The Banach technique has two main ingredients: a complete metric space and a contractive self-mapping from the metric space into itself. Searching for more general results, the sense in which a mapping can be considered contractive has changed throughout the last ninety years. In the original version, an operator $T : X \to X$ from a metric space (X, d) into itself is contractive if there exists a

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constant $\lambda \in [0,1)$ such that $d(Tx,Ty) \leq \lambda\, d(x,y)$. In particular, when $x \neq y$, the strict inequality $d(Tx,Ty) < d(x,y)$ follows from the contractivity condition. In general, if there exist $x_0, y_0 \in X$, with $x_0 \neq y_0$, such that $d(Tx_0,Ty_0) = d(x_0,y_0)$, the existence and uniqueness of fixed points is not ensured (for instance, consider a translation in the set of all real numbers).

Although weaker contractivity conditions have appeared, especially involving several classes of control functions (see, e.g., [**1, 5, 7, 12, 15**]), the existence of two different points for which the operator acts as an isometry is usually forbidden by most contractivity assumptions. In order to face this problem, some contractivity requirements were introduced in the past, using in the right-hand side of the contractivity condition a term that can be greater than $d(x,y)$. In this sense, rational type contractivity conditions (as the following ones) have played an important role. We must highlight that many of them are not symmetric in their variables.

In [**11**], Dass and Gupta presented the following fixed point theorem.

Theorem 1 (Dass and Gupta [**11**])**.** *Let (X,d) be a complete metric space and $T : X \to X$ a mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx,Ty) \leq \frac{\alpha d(y,Ty)(1 + d(x,Tx))}{1 + d(x,y)} + \beta d(x,y)$$

for all $x, y \in X$. Then T has a unique fixed point.

Recently, in [**10**], the authors proved the following result.

Theorem 2 (Can and Thuan [**10**])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping satisfying*

$$\varphi(d(Tx,Ty)) \leq \varphi(M(x,y)) - \psi(M(x,y)) \quad \text{for all } x, y \in X,$$

where $M(x,y)$ is given by

$$M(x,y) = \max\left(d(x,y),\ \frac{d(y,Ty)(1 + d(x,Tx))}{1 + d(x,y)} \right)$$

and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and monotone non-decreasing function with $\varphi(t) = 0$ if and only if $t = 0$ and $\psi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\psi(t) = 0$ if and only if $t = 0$.

Then T has a unique fixed point.

In the context of ordered metric spaces, the following result was published in [**9**].

Theorem 3 (Cabrera, Harjani and Sadarangani [**9**])**.** *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that there exists $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx,Ty) \leq \frac{\alpha d(y,Ty)(1+d(x,Tx))}{1+d(x,y)} + \beta d(x,y) \quad \textit{for all } x,y \in X \textit{ with } x \preccurlyeq y.$$

If there exists $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$*, then* T *has a fixed point.*

Notice that Theorem 3 is Theorem 1 in the context of ordered metric spaces.

In Theorems 1, 2 and 3, the contractivity condition is not symmetric in the sense that, if we change x by y, the contractivity condition changes. From a practical point of view, it is convenient that these contractivity conditions are symmetric.

This paper has two main purposes: as first objective, we present a generalization of Theorems 1, 2 and 3 by using different kinds of control functions both in the setting of metric spaces and in the framework of partially ordered metric spaces. As second objective, we highlight that the lack of symmetry in the kind of rational type contractivity conditions can be overcome. Some examples, involving continuous and non-continuous mappings, are given to illustrate the fact that the presented statements are applicable when the nonlinear operator acts as an isometry over some points. As an application, in the last section, we show how to employ our main result in order to solve a problem which appears in dynamic programming.

2. PRELIMINARIES

In [**13**], Matkowski considered functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following properties (where ϕ^n denotes the n-th iteration of ϕ).

($\mathcal{P}_1$) ϕ is non-decreasing.

($\mathcal{P}_2$) $\{\phi^n(t)\} \to 0$ for all $t > 0$.

($\mathcal{P}_3$) $\phi(t) < t$ for all $t > 0$.

Although conditions ($\mathcal{P}_2$) and ($\mathcal{P}_3$) are, in general, independent, under hypothesis ($\mathcal{P}_1$), we have the following result.

Proposition 4 (Matkowski [**13**]). $(\mathcal{P}_1) + (\mathcal{P}_2) \Rightarrow (\mathcal{P}_3)$.

Proof. By contradiction, assume that there exists $t_0 > 0$ such that $t_0 \leq \phi(t_0)$. As ϕ is non-decreasing, then $\phi(t_0) \leq \phi(\phi(t_0))$, which implies that $t_0 \leq \phi(t_0) \leq \phi^2(t_0)$. By induction, it can be proved that $t_0 \leq \phi^n(t_0)$ for all $n \in \mathbb{N}$. Hence, $\{\phi^n(t_0)\}$ cannot converge to zero, which is a contradiction. □

Although functions verifying ($\mathcal{P}_1$) and ($\mathcal{P}_2$) (and, consequently, also ($\mathcal{P}_3$)) should be called *Matkowski functions*, over time, these functions, due to their applications, are known as *comparison functions* (see, for instance, [**5, 4, 6**]).

Definition 5. *A* comparison function *is a non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\{\phi^n(t)\} \to 0$ for all $t > 0$. Let $\mathcal{F}_{\text{com}}$ denote the family of all comparison functions.*

In [**2**], the authors proved the following result.

Proposition 6. *For a non-decreasing, upper semi-continuous function $\phi : [0,\infty) \to [0,\infty)$, properties $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$ are equivalent.*

In particular, if ϕ is continuous and non-decreasing, then ϕ is a comparison function if, and only if, $\phi(t) < t$ for all $t > 0$. This remark proves, as a consequence of the mean value theorem, that the following are examples of comparison functions.

- $\phi(t) = \lambda t$ (where $\lambda \in [0,1)$);
- $\phi(t) = \arctan t$;
- $\phi(t) = \ln(1+t)$;
- $\phi(t) = \dfrac{t}{1+t}$.

In this paper, we denote by $\mathbb{N}$ the set of all non-negative integer numbers $\{0,1,2,\ldots\}$.

3. MAIN RESULTS

Our first main result is the following.

Theorem 7. *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping. Assume that there exist $L \geq 0$ and a continuous $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$d(Tx,Ty) \leq \phi\left(\max\left\{ d(x,y), \frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)}, \frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)} \right\} \right) + L \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{1}$$

for all $x,y \in X$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}$ be the sequence defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, so x_{n_0} is a fixed point of T. In this case, the proof is finished. Suppose, on the contrary, that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, that is,

$$d(x_n, x_{n+1}) > 0 \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Applying the contractivity condition (1), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\leq \phi\Bigg(\max\Bigg\{ d(x_n, x_{n+1}), \frac{d(x_n,Tx_n)(1+d(x_{n+1},Tx_{n+1}))}{1+d(x_n,x_{n+1})}, \\ &\qquad \frac{d(x_{n+1},Tx_{n+1})(1+d(x_n,Tx_n))}{1+d(x_n,x_{n+1})} \Bigg\} \Bigg) \end{aligned}$$

$$
\begin{aligned}
&+ L\min\left\{d(x_n,Tx_n), d(x_{n+1},Tx_{n+1}), d(x_n,Tx_{n+1}), d(x_{n+1},Tx_n)\right\}\\
&\le \phi\Bigg(\max\Bigg\{d(x_n,x_{n+1}), \frac{d(x_n,x_{n+1})(1+d(x_{n+1},x_{n+2}))}{1+d(x_n,x_{n+1})},\\
&\qquad\qquad \frac{d(x_{n+1},x_{n+2})(1+d(x_n,x_{n+1}))}{1+d(x_n,x_{n+1})}\Bigg\}\Bigg)\\
&+ L\min\left\{d(x_n,Tx_n), d(x_{n+1},x_{n+2}), d(x_n,x_{n+2}), 0\right\}
\end{aligned}
$$

$$
(3) = \phi\left(\max\left\{d(x_n,x_{n+1}), \frac{d(x_n,x_{n+1})(1+d(x_{n+1},x_{n+2}))}{1+d(x_n,x_{n+1})}, d(x_{n+1},x_{n+2})\right\}\right)
$$

Consider the subsets

$$N_1 = \{\, n\in\mathbb{N}:\ \text{the maximum in (3) is}\ \ d(x_n,x_{n+1})\,\},$$

$$N_2 = \left\{\, n\in\mathbb{N}:\ \text{the maximum in (3) is}\ \ \frac{d(x_n,x_{n+1})\ (1+d(x_{n+1},x_{n+2}))}{1+d(x_n,x_{n+1})}\right\},$$

$$N_3 = \{\, n\in\mathbb{N}:\ \text{the maximum in (3) is}\ \ d(x_{n+1},x_{n+2})\,\}.$$

Clearly $N_1\cup N_2\cup N_3=\mathbb{N}$. We claim that

$$d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1}) \quad \text{for all } n\in\mathbb{N}. \tag{4}$$

We consider the following cases.

- If $n\in N_1$, then, by $(\mathcal{P}_3)$ and (2), we have that $d(x_{n+1},x_{n+2})\le\phi(d(x_n,x_{n+1})) < d(x_n,x_{n+1})$, so (4) holds.
- If $n\in N_2$, then, using $(\mathcal{P}_3)$ and (2) again,

$$
\begin{aligned}
d\left(x_{n+1},x_{n+2}\right) &\le \phi\left(\frac{d(x_n,x_{n+1})\ (1+d(x_{n+1},x_{n+2}))}{1+d(x_n,x_{n+1})}\right)\\
&< \frac{d(x_n,x_{n+1})\ (1+d(x_{n+1},x_{n+2}))}{1+d(x_n,x_{n+1})}.
\end{aligned}
$$

  Hence

$$
\begin{aligned}
&d\left(x_{n+1},x_{n+2}\right) + d(x_n,x_{n+1})\, d\left(x_{n+1},x_{n+2}\right)\\
&\qquad < d(x_n,x_{n+1}) + d(x_n,x_{n+1})\, d(x_{n+1},x_{n+2}),
\end{aligned}
$$

  which yields $d\left(x_{n+1},x_{n+2}\right) < d(x_n,x_{n+1})$, that is, (4) holds.
- If $n\in N_3$, then $d(x_{n+1},x_{n+2})\le\phi(d(x_{n+1},x_{n+2})) < d(x_{n+1},x_{n+2})$, which is impossible.

In any case, we proved that (4) holds. Since $\{d(x_n,x_{n+1})\}$ is a non-increasing sequence of non-negative real numbers, it is convergent. Let $a\ge 0$ be its limit. As ϕ is continuous, letting $n\to\infty$ in (3), we deduce that

$$a \le \phi\left(\max\left\{a, \frac{a(1+a)}{1+a}, a\right\}\right) = \phi(a),$$

which implies that $a = 0$, that is, $\{d(x_n, x_{n+1})\} \to 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in (X, d). Reasoning by contradiction, suppose the contrary case. Then, it is well known (see, for instance, [**7, 14**]) that there exist $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,

$$k \leq n(k) < m(k), \quad d(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon_0 < d(x_{n(k)}, x_{m(k)})$$

and also

$$\begin{aligned}(5) \qquad \lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) &= \lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)}) \\ &= \lim_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \varepsilon_0.\end{aligned}$$

Applying the contractivity condition (1), it follows that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&d(x_{n(k)+1}, x_{m(k)+1}) = d(Tx_{n(k)}, Tx_{m(k)}) \\
&\quad \leq \phi\bigg(\max\Big\{ d(x_{n(k)}, x_{m(k)}), \frac{d(x_{n(k)}, Tx_{n(k)})(1 + d(x_{m(k)}, Tx_{m(k)}))}{1 + d(x_{n(k)}, x_{m(k)})}, \\
&\qquad\qquad \frac{d(x_{m(k)}, Tx_{m(k)})(1 + d(x_{n(k)}, Tx_{n(k)}))}{1 + d(x_{n(k)}, x_{m(k)})} \Big\} \bigg) \\
(6) \quad &\qquad + L \min\big\{ d(x_{n(k)}, Tx_{n(k)}), \ d(x_{m(k)}, Tx_{m(k)}), \\
&\qquad\qquad d(x_{n(k)}, Tx_{m(k)}), d(x_{m(k)}, Tx_{n(k)}) \big\} \\
&\quad = \phi\bigg(\max\Big\{ d(x_{n(k)}, x_{m(k)}), \frac{d(x_{n(k)}, x_{n(k)+1})(1 + d(x_{m(k)}, x_{m(k)+1}))}{1 + d(x_{n(k)}, x_{m(k)})}, \\
&\qquad\qquad \frac{d(x_{m(k)}, x_{m(k)+1})(1 + d(x_{n(k)}, x_{n(k)+1}))}{1 + d(x_{n(k)}, x_{m(k)})} \Big\} \bigg) \\
&\quad + L \min\big\{ d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{m(k)+1}), d(x_{m(k)}, x_{n(k)+1}) \big\}.
\end{aligned}$$

Taking into account that ϕ is continuous and (5), and letting $n \to \infty$ in (6), we deduce that

$$\varepsilon_0 \leq \phi\left(\max\{\varepsilon_0, 0, 0\}\right) + L \min\{0, 0, \varepsilon_0, \varepsilon_0\} = \phi(\varepsilon_0) < \varepsilon_0,$$

which is a contradiction. As a consequence, $\{x_n\}$ is a Cauchy sequence in (X, d). Since it is complete, then there exists $u \in X$ such that $\{x_n\} \to u$. We claim that u is a fixed point of T. Indeed, by using the contractivity condition (1), it follows that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&d(x_{n+1}, Tu) = d(Tx_n, Tu) \\
&\leq \phi\bigg(\max\Big\{ d(x_n, u), \frac{d(x_n, Tx_n)(1 + d(u, Tu))}{1 + d(x_n, u)}, \frac{d(u, Tu)(1 + d(x_n, Tx_n))}{1 + d(x_n, u)} \Big\} \bigg) \\
&\qquad + L \min\big\{ d(x_n, Tx_n), d(u, Tu), d(x_n, Tu), d(u, Tx_n) \big\}
\end{aligned}$$

$$= \phi\Big(\max\Big\{ d(x_n, u), \frac{d(x_n, x_{n+1})(1 + d(u, Tu))}{1 + d(x_n, u)}, \frac{d(u, Tu)(1 + d(x_n, x_{n+1}))}{1 + d(x_n, u)} \Big\}\Big) + L \min\big\{ d(x_n, x_{n+1}), d(u, Tu), d(x_n, Tu), d(u, x_{n+1}) \big\}.$$

Since $\{x_n\} \to u$, letting $n \to \infty$ in the previous inequality, we deduce that

$$d(u, Tu) \le \phi\Big(\max\Big\{ 0, \frac{0(1 + d(u, Tu))}{1 + 0}, \frac{d(u, Tu)(1 + 0)}{1 + 0} \Big\}\Big) + L \min\big\{ 0, d(u, Tu), d(u, Tu), 0 \big\} = \phi\big(d(u, Tu)\big),$$

which means that $d(u, Tu) = 0$, that is, $Tu = u$ and u is a fixed point of T.

Next, we prove that u is the unique fixed point of T. Let x and y be arbitrary fixed points of T. Using the contractivity condition (1), it follows that

$$\begin{aligned} d(x, y) &= d(Tx, Ty) \\ &\le \phi\Big(\max\Big\{ d(x, y), \frac{d(x, Tx)(1 + d(y, Ty))}{1 + d(x, y)}, \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)} \Big\}\Big) \\ &\quad + L \min\big\{ d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \big\} \\ &= \phi\Big(\max\Big\{ d(x, y), \frac{0(1 + 0)}{1 + d(x, y)}, \frac{0(1 + 0)}{1 + d(x, y)} \Big\}\Big) + L \min\{0, 0, d(x, y), d(y, x)\} \\ &= \phi\left(d(x, y)\right), \end{aligned}$$

which means that $d(x, y) = 0$. Thus, $x = y$ and T has a unique fixed point. Therefore, the proof is complete. □

The following example shows that Theorem 7 is applicable when other results fail.

Example 8. Let X be the subset of real numbers $[-2, 2] \cup \{3, 4\}$ endowed with the Euclidean metric $d(x, y) = |\, x - y \,|$ for all $x, y \in X$. Since X is closed in $\mathbb{R}$, then (X, d) is a complete metric space. Let $T : X \to X$ be the self-mapping given, for all $x \in X$, by

$$Tx = \begin{cases} x/2, & \text{if } x \in [-2, 2], \\ 1, & \text{if } x = 3, \\ 2, & \text{if } x = 4. \end{cases}$$

Notice that the points $z_0 = 3$ and $\omega_0 = 4$ verify the equality $d(Tz_0, T\omega_0) = d(z_0, \omega_0)$. Therefore, the classical Banach theorem and many other generalizations are not applicable. However, we claim that the operator T satisfies the contractivity condition (1) using $\phi_{1/2} \in \mathcal{F}_{\mathrm{com}}$ and $L = 0$, where $\phi_{1/2}(t) = t/2$ for all $t \in [0, \infty)$. Indeed, let $x, y \in X$ be arbitrary points. Suppose, without loss of generality, that $x \le y$.

- If $x, y \in [-2, 2]$, then it follows that $d(Tx, Ty) = |\, x/2 - y/2 \,| = (1/2)\ |\, x - y \,| = \phi_{1/2}(d(x, y))$.
- If $x \in [-2, 2]$ and $y = 3$, then $d(Tx, Ty) = |x/2 - 1| = 1 - x/2 = (1/2)\,(2 - x) \le (1/2)\ (3 - x) = \phi_{1/2}(d(x, y))$.

- If $x \in [-2,2]$ and $y = 4$, then $d(Tx,Ty) = |\,x/2 - 2\,| = 2 - x/2 = (1/2)\,(4-x) = \phi_{1/2}(d(x,y))$.
- Finally, if $x = 3$ and $y = 4$, then $d(T3,T4) = d(1,2) = 1 = d(3,4)$. Therefore, the condition $d(T3,T4) \leq \lambda\, d(3,4)$ is not met, for $\lambda \in [0,1)$. However, in this case,

$$\phi_{1/2}\left(\frac{d(x,Tx)\ \ (1+d(y,Ty))}{1+d(x,y)}\right) = \frac{1}{2}\cdot\frac{d(3,1)\ \ (1+d(4,2))}{1+d(3,4)}$$
$$= \frac{1}{2}\cdot\frac{2\ (1+2)}{1+1} = \frac{3}{2} > 1 = d(T3,T4).$$

As a consequence, the contractivity condition (1) holds in any case. Therefore, applying Theorem 7, T has a unique fixed point (which is $u = 0$).

In the following example, T is not continuous, but the same conclusion is valid.

Example 9. Let X be the interval of real numbers $[-2,3]$ endowed with the Euclidean metric $d(x,y) = |\,x-y\,|$ for all $x,y \in X$. As X is closed in $\mathbb{R}$, then (X,d) is a complete metric space. Let $T : X \to X$ be the self-mapping given, for all $x \in X$, by

$$Tx = \begin{cases} x/2, & \text{if } x \in [-2,2)\,, \\ 0, & \text{if } x \in [2,3)\,, \\ 1, & \text{if } x = 3. \end{cases}$$

Clearly T is not continuous at $x = 2$ and $x = 3$. Furthermore, the points $z_0 = 2$ and $\omega_0 = 3$ verify the equality $d(Tz_0,T\omega_0) = d(z_0,\omega_0)$. Therefore, many results in the field of fixed point theory are not applicable. However, the operator T satisfies the contractivity condition (1) using $\phi_{1/2} \in \mathcal{F}_{\rm com}$ and $L = 0$, where $\phi_{1/2}(t) = t/2$ for all $t \in [0,\infty)$. Therefore, applying Theorem 7, T has a unique fixed point (which is $u = 0$).

In order to present a new example, we need the following proposition.

Proposition 10. *The following inequalities hold.*

1. $4.5x + 4y \leq 15 + xy$ *for all* $x \in [0,2]$ *and all* $y \in [2,3]$.
2. $x^2 + y^2 + 2x + 3y \leq 4xy + 3$ *for all* $x,y \in [2,3]$.
3. $x - x^2 \leq y(1 - x/2)$ *for all* $x \in [0,2]$ *and all* $y \in [3,\infty)$.
4. $4y - 4x + x^2 + 3 \leq 2xy$ *for all* $x \in [2,3]$ *and all* $y \in [3,\infty)$.

In the following example, (X,d) is not bounded, and other previous results (like Boyd and Wong's theorem [**8**] or Matkowski's theorem) cannot be applied.

Example 11. Let $X = \mathbb{R}$ be the set of all real numbers endowed with the complete Euclidean metric $d(x,y) = |\,x-y\,|$ for all $x,y \in X$, Let $T : X \to X$ be the self-mapping given, for all $x \in X$, by

$$Tx = \begin{cases} x/2, & \text{if } x \in [0,2]\,, \\ 3-x, & \text{if } x \in (2,3]\,, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, T is continuous and its image is included in $[0,1]$. Hence, $d(Tx,Ty) \leq 1$ for all $x,y \in X$. We claim that T satisfies the contractivity condition (1) using $\phi_{1/2} \in \mathcal{F}_{\text{com}}$ and $L=0$, where $\phi_{1/2}(t) = t/2$ for all $t \in [0,\infty)$. Let $x,y \in X$ be arbitrary. Suppose, without loss of generality, that $x<y$. To prove the claim, we distinguish the following cases.

(C1) If $d(Tx,Ty) \leq \phi_{1/2}\left(d(x,y)\right)$, then (1) holds. As a consequence, in the subsequent cases, we can assume that

$$0 < \frac{y-x}{2} = \phi_{1/2}\left(d(x,y)\right) < d(Tx,Ty).$$

In particular, $Tx \neq Ty$.

(C2) If $d(x,y) \geq 2$, then

$$d(Tx,Ty) \leq 1 = \frac{1}{2}\,2 \leq \frac{1}{2}\,d(x,y) = \phi_{1/2}\left(d(x,y)\right),$$

which reduces to the case (C1). As a consequence, in the subsequent cases, we can assume that

$$0 < y-x < 2.$$

In this case, notice that

$$\frac{1}{3} < \frac{1}{1+y-x} < 1. \tag{7}$$

- If $x \leq 0$ and $y \in [0,2]$, then

$$\begin{aligned} d(Tx,Ty) &= d(0,y/2) = \frac{y}{2} \leq \frac{y+|x|}{2} = \frac{y-x}{2} = \phi_{1/2}\left(y-x\right) \\ &= \phi_{1/2}\left(d(x,y)\right), \end{aligned}$$

which reduces to case (C1).

- If $x,y \in [0,2]$, then case (C1) can be applied.
- If $x \in [0,2]$ and $y \in (2,3]$, then

$$\begin{aligned} d(Tx,Ty) &= d(x/2,3-y) = \left|\,\frac{x}{2}-(3-y)\,\right| = \frac{|\,x+2y-6\,|}{2} \quad \text{and} \\ \phi_{1/2}\left(\frac{d(y,Ty)\,\left(1+d(x,Tx)\right)}{1+d(x,y)}\right) &= \frac{1}{2}\,\frac{|\,y-(3-y)\,|\,\left(1+|\,x-x/2\,|\right)}{1+|\,x-y\,|} \\ &= \frac{1}{2}\,\frac{(2y-3)\,\left(1+x/2\right)}{1+y-x} = \frac{2y+xy-\frac{3}{2}x-3}{2\left(1+y-x\right)}. \end{aligned}$$

We have the following subcases, depending on the sign of $x+2y-6$.

▶ Assume that $x+2y-6 \geq 0$. Let show that

$$d(Tx,Ty) \leq \phi_{1/2}\left(\frac{d(y,Ty)\,\left(1+d(x,Tx)\right)}{3}\right).$$

Indeed, notice that

$$d(Tx,Ty) \leq \phi_{1/2}\left(\frac{d(y,Ty)\,\left(1+d(x,Tx)\right)}{3}\right)$$

$$\Leftrightarrow \frac{x+2y-6}{2} \leq \frac{2y+xy-\frac{3}{2}x-3}{2\cdot 3}$$
$$\Leftrightarrow 3x+6y-18 \leq 2y+xy-1.5x-3 \Leftrightarrow 4.5x+4y \leq 15+xy,$$

which is true by item 10 of Proposition 10. Hence, by (7), we deduce that

$$d(Tx,Ty) \leq \phi_{1/2}\Big(\frac{d(y,Ty)\ (1+d(x,Tx))}{3}\Big) \leq \phi_{1/2}\Big(\frac{d(y,Ty)\ (1+d(x,Tx))}{1+d(x,y)}\Big),$$

which means that (1) holds.

► Assume that $x+2y-6<0$. Then

$$d(Tx,Ty) \leq \phi_{1/2}\,(d(x,y)) \quad \Leftrightarrow \quad \frac{6-x-2y}{2} \leq \frac{y-x}{2} \quad \Leftrightarrow \quad 6 \leq 3y,$$

which is true because $y \geq 2$, so case (C1) is applicable.

- If $x, y \in [2,3]$, then

$$d(Tx,Ty) = |\,(3-x)-(3-y)\,| = |\,x-y\,| = y-x \quad \text{and}$$
$$\phi_{1/2}\left(\frac{d(y,Ty)\ (1+d(x,Tx))}{1+d(x,y)}\right) = \frac{1}{2}\,\frac{|\,y-(3-y)\,|\ (1+|\,x-(3-x)\,|)}{1+|\,x-y\,|}$$
$$= \frac{1}{2}\,\frac{(2y-3)\ (2x-2)}{1+y-x} = \frac{2xy-2y-3x+3}{1+y-x}.$$

Therefore,

$$d(Tx,Ty) \leq \phi_{1/2}\left(\frac{d(y,Ty)\ (1+d(x,Tx))}{1+d(x,y)}\right)$$
$$\Leftrightarrow \quad y-x \leq \frac{2xy-2y-3x+3}{1+y-x}$$
$$\Leftrightarrow \quad x^2-2xy-x+y^2+y \leq 2xy-2y-3x+3$$
$$\Leftrightarrow \quad x^2+y^2+2x+3y \leq 4xy+3.$$

This inequality holds by item 10 of Proposition 10.

- If $x \in [0,2]$ and $y \geq 3$, then

$$d(Tx,Ty) = \left|\,\frac{x}{2}-0\,\right| = \frac{x}{2} \quad \text{and}$$
$$\phi_{1/2}\left(\frac{d(y,Ty)\ (1+d(x,Tx))}{1+d(x,y)}\right) = \frac{1}{2}\,\frac{|\,y-0\,|\ (1+|\,x-x/2\,|)}{1+|\,x-y\,|}$$
$$= \frac{y\,+xy/2}{2\,(1+y-x)}.$$

Hence,

$$d(Tx,Ty) \leq \phi_{1/2}\left(\frac{d(y,Ty)\ (1+d(x,Tx))}{1+d(x,y)}\right)$$
$$\Leftrightarrow \quad \frac{x}{2} \leq \frac{y\,+xy/2}{2\,(1+y-x)}$$
$$\Leftrightarrow \quad x+xy-x^2 \leq y\,+xy/2 \quad \Leftrightarrow \quad x-x^2 \leq y(1-x/2).$$

This inequality holds by item 10 of Proposition 10.

- If $x \in [2,3]$ and $y \geq 3$, then

$$\begin{aligned} d(Tx,Ty) &= |\,(3-x)-0\,| = 3-x \quad \text{and} \\ \phi_{1/2}\left(\frac{d(y,Ty)\ (1+d(x,Tx))}{1+d(x,y)}\right) &= \frac{1}{2}\,\frac{|\,y-0\,|\ (1+|\,x-(3-x)\,|)}{1+|\,x-y\,|} \\ &= \frac{y\,(1+|\,2x-3\,|)}{2\,(1+y-x)} = \frac{y\,(2x-2)}{2\,(1+y-x)} = \frac{y\,(x-1)}{1+y-x}. \end{aligned}$$

Therefore,

$$\begin{aligned} d(Tx,Ty) &\leq \phi_{1/2}\left(\frac{d(y,Ty)\ (1+d(x,Tx))}{1+d(x,y)}\right) \\ &\Leftrightarrow \quad 3-x \leq \frac{y\,(x-1)}{1+y-x} \\ &\Leftrightarrow \quad 3y-4x-xy+x^2+3 \leq xy-y \quad \Leftrightarrow \quad 4y-4x+x^2+3 \leq 2xy. \end{aligned}$$

This inequality holds by item 10 of Proposition 10.

As a consequence of the previous cases, T satisfies the contractivity condition (1). Hence, Theorem 7 guarantees that T has a unique fixed point (which is $x=0$).

Remark 12. Notice that in the previous example, if $x, y \in [2,3]$, then $d(Tx,Ty) = d(x,y)$. Thus, we cannot apply any result in which the contractivity condition is of the type

$$d(Tx,Ty) \leq \phi(d(x,y))$$

where $\phi : [0,\infty) \to [0,\infty)$ satisfies the condition $\phi(t) < t$ for all $t > 0$. For instance, we cannot use Boyd and Wong's theorem nor Matkowski's theorem.

As we have seen in the proof of Theorem 7, the role of the terms $d(x,Ty)$ and $d(y,Tx)$ in the summand

$$\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

in the contractivity condition (1) is very important. Without such terms, Theorem 7 would not hold, as we can see in the following example.

Example 13. Let $\mathcal{C}([0,\pi],\mathbb{R})$ the family of all continuous functions $f : [0,\pi] \to \mathbb{R}$, and let d_∞ be the complete metric on $\mathcal{C}([0,\pi],\mathbb{R})$ given by

$$d_\infty(f,g) = \sup\left(\{|\,f(x)-g(x)\,| : x \in [0,\pi]\}\right) \quad \text{for all } f,g \in \mathcal{C}([0,\pi],\mathbb{R}).$$

For each $\alpha \in [0,1]$, let $f_\alpha : [0,\pi] \to \mathbb{R}$ be the function given by $f_\alpha(x) = 1 - \alpha \sin x$ for all $x \in [0,\pi]$. Then $f_\alpha \in \mathcal{C}([0,\pi],\mathbb{R})$ for all $\alpha \in [0,1]$. Let $A = \{f_\alpha : \alpha \in [0,1]\}$. Taking into account that, for all $\alpha, \beta \in [0,1]$,

$$\begin{aligned} d_\infty(f_\alpha, f_\beta) &= \sup\left(\{|\,(1-\alpha \sin x)-(1-\beta \sin x)\,| : x \in [0,\pi]\}\right) \\ &= \sup\left(\{|\,\alpha-\beta\,|\,\sin x : x \in [0,\pi]\}\right) = |\,\alpha-\beta\,|, \end{aligned}$$

then A is a closed, complete subset of $\mathcal{C}([0,\pi],\mathbb{R})$. Next, let $B = \{-f_\alpha : \alpha \in [0,1]\}$ and $X = A \cup B$. The same reasoning shows that B is also a closed, complete subset of $\mathcal{C}([0,\pi],\mathbb{R})$, so (X, d_∞) is a complete metric space. Let $T : X \to X$ be the self-mapping given by $T(f) = -f$ for all $f \in X$. Clearly, T has not any fixed point. However, we claim that T satisfies that

$$d_\infty(Tx,Ty) \leq \min\{\ d_\infty(x,Tx),\ d_\infty(y,Ty)\ \} \quad \text{for all } x,y \in X.$$

Indeed, notice that, for all $\alpha \in [0,1]$,

$$\begin{aligned} d(f_\alpha, T(f_\alpha)) &= d_\infty(f_\alpha, -f_\alpha) \\ &= \sup\left(\{\,|\,(1-\alpha\,\sin x) - (-1+\alpha\,\sin x)\,|\,:x\in[0,\pi]\}\right) \\ &= \sup\left(\{\,|\,2-2\alpha\,\sin x\,|\,:x\in[0,\pi]\}\right) \\ &= \sup\left(\{\,2-2\alpha\,\sin x\ :x\in[0,\pi]\}\right) = 2, \end{aligned}$$

and similarly, $d_\infty(-f_\alpha, T(-f_\alpha)) = d_\infty(-f_\alpha, f_\alpha) = 2$. As a result,

$$d_\infty(x,Tx) = 2 \quad \text{for all } x \in X.$$

On the other hand, for all $\alpha,\beta \in [0,1]$,

$$d(T(f_\alpha),T(f_\beta)) = d_\infty(-f_\alpha,-f_\beta) = d_\infty(f_\alpha,f_\beta) = |\,\alpha-\beta\,| \leq 1,$$

$$d(T(-f_\alpha),T(f_\beta)) = d_\infty(f_\alpha,-f_\beta) = 2,$$

$$d(T(-f_\alpha),T(-f_\beta)) = d_\infty(f_\alpha,f_\beta) = |\,\alpha-\beta\,| \leq 1.$$

Hence, for all $x,y \in X$,

$$d_\infty(Tx,Ty) \leq 2 = \min\{\ d_\infty(x,Tx),\ d_\infty(y,Ty)\ \}\,.$$

As a consequence, T satisfies the inequality

$$\begin{aligned} d_\infty(Tx,Ty) \leq \phi\bigg(&\max\Big\{\ d_\infty(x,y),\ \frac{d_\infty(x,Tx)\ (1+d_\infty(y,Ty))}{1+d_\infty(x,y)},\frac{d_\infty(y,Ty)(1+d_\infty(x,Tx))}{1+d_\infty(x,y)}\Big\}\bigg) \\ &+ L\min\{d_\infty(x,Tx),d_\infty(y,Ty)\} \end{aligned}$$

for all $x,y \in X$. Notice that T does not have any fixed point. This means that the role of the terms $d_\infty(x,Ty)$ and $d_\infty(y,Tx)$ in the contractivity condition (1) is very important, because

$$\min\{d_\infty(x,Tx),d_\infty(y,Ty),d_\infty(x,Ty),d_\infty(y,Tx)\} \leq \min\{d_\infty(x,Tx),d_\infty(y,Ty)\}$$

for all $x,y \in X$.

Some consequences of Theorem 7 are the following results.

Corollary 14. *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \leq \phi\left(\max\left\{d(x,y),\frac{d(x,Tx)\ (1+d(y,Ty))}{1+d(x,y)},\frac{d(y,Ty)\ (1+d(x,Tx))}{1+d(x,y)}\right\}\right)$$

for all $x,y \in X$, where ϕ is a continuous comparison function. Then T has a unique fixed point.

Corollary 15. *Let (X,d) be a complete metric space and let $T:X\to X$ be a mapping. Assume that there exist $\alpha,\beta,\gamma,L\geq 0$, with $\alpha+\beta+\gamma<1$, such that*

$$d(Tx,Ty)\leq \alpha\, d(x,y)+\beta\,\frac{d(x,Tx)\ (1+d(y,Ty))}{1+d(x,y)}+\gamma\,\frac{d(y,Ty)\ (1+d(x,Tx))}{1+d(x,y)}$$

$$+\,L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$$

for all $x,y\in X$. Then T has a unique fixed point.

Proof. Let $\lambda=\alpha+\beta+\gamma\in[0,1)$ and let $\phi_\lambda:[0,\infty)\to[0,\infty)$ be the function given by $\phi_\lambda(t)=\lambda t$ for all $t\in[0,\infty)$. As $\lambda<1$, then ϕ_λ is a continuous comparison function. Furthermore, for all $x,y\in X$,

$$\begin{aligned}
d(Tx,Ty) &\leq \alpha\, d(x,y)+\beta\frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)}+\gamma\frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}\\
&\quad + L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}\\
&\leq \alpha\max\left\{d(x,y),\frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)},\frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}\right\}\\
&\quad +\beta\max\left\{d(x,y),\frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)},\frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}\right\}\\
&\quad +\gamma\max\left\{d(x,y),\frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)},\frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}\right\}\\
&\quad + L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}\\
&= \phi_\lambda\left(\max\left\{d(x,y),\frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)},\frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}\right\}\right)\\
&\quad + L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}\,.
\end{aligned}$$

Hence, Theorem 7 guarantees that T has a unique fixed point. □

Notice that Corollary 15 is a generalization of Theorem 1.

Using similar arguments, we can obtain the following generalization of Theorems 2 and 3, in which we use the following notation.

$$\begin{aligned}
\mathcal{F}_{\text{alt}} &= \{\,\phi:[0,\infty)\to[0,\infty):\ \phi \text{ continuous, non-decreasing, } \phi\,(t)=0\Leftrightarrow t=0\,\}\,,\\
\mathcal{F}'_{\text{alt}} &= \{\,\phi:[0,\infty)\to[0,\infty):\ \phi \text{ lower semi-continuous, } \phi\,(t)=0\Leftrightarrow t=0\,\}\,.
\end{aligned}$$

Theorem 16. *Let (X,d) be a complete metric space and let $T:X\to X$ be a mapping. Assume that there exist $\varphi\in\mathcal{F}_{\text{alt}}$ and $\psi\in\mathcal{F}'_{\text{alt}}$ such that*

$$\varphi(d(Tx,Ty))\leq\varphi(N(x,y))-\psi\,(N(x,y)) \tag{8}$$

for all $x,y\in X$, where

$$N(x,y)=\max\left\{d(x,y),\frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)},\frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}\right\}.$$

Then T has a unique fixed point.

Proof. First of all, we show that, for $t, s \in [0, \infty)$,

$$s > 0, \quad \varphi(t) \leq \varphi(s) - \psi(s) \quad \Rightarrow \quad t < s. \tag{9}$$

Indeed, assume that $t \geq s$. As φ is nondecreasing,

$$\varphi(s) \leq \varphi(t) \leq \varphi(s) - \psi(s) \leq \varphi(s),$$

which implies that $\psi(s) = 0$. Taking into account that $\psi^{-1}(\{0\}) = \{0\}$, we deduce that $s = 0$, which contradicts the fact that $s > 0$. Thus, $t < s$ and (9) holds.

Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}$ be the sequence defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, so x_{n_0} is a fixed point of T. In this case, the proof is finished. Suppose, on the contrary, that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, that is,

$$d(x_n, x_{n+1}) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Applying the contractivity condition (8), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \varphi\left(d\left(x_{n+1}, x_{n+2}\right)\right) &= \varphi\left(d\left(Tx_n, Tx_{n+1}\right)\right) \\ &\leq \varphi(N(x_n, x_{n+1})) - \psi\left(N(x_n, x_{n+1})\right), \end{aligned} \tag{10}$$

where

$$\begin{aligned} N(x_n, x_{n+1}) &= \max\left\{d(x_n, x_{n+1}), \frac{d(x_n, Tx_n)(1 + d(x_{n+1}, Tx_{n+1}))}{1 + d(x_n, x_{n+1})}, \right. \\ &\qquad \left. \frac{d(x_{n+1}, Tx_{n+1})(1 + d(x_n, Tx_n))}{1 + d(x_n, x_{n+1})}\right\} \\ &\leq \max\left\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1})(1 + d(x_{n+1}, x_{n+2}))}{1 + d(x_n, x_{n+1})}, \right. \\ &\qquad \left. \frac{d(x_{n+1}, x_{n+2})(1 + d(x_n, x_{n+1}))}{1 + d(x_n, x_{n+1})}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1})(1 + d(x_{n+1}, x_{n+2}))}{1 + d(x_n, x_{n+1})}, d(x_{n+1}, x_{n+2})\right\}. \end{aligned} \tag{11}$$

Notice that $N(x_n, x_{n+1}) \geq d(x_n, x_{n+1}) > 0$. By (9) and (10),

$$\begin{aligned} d\left(x_{n+1}, x_{n+2}\right) < N\left(x_n, x_{n+1}\right) = \max\bigg\{ & d(x_n, x_{n+1}),\ d(x_{n+1}, x_{n+2}), \\ & \frac{d(x_n, x_{n+1})\ (1 + d(x_{n+1}, x_{n+2}))}{1 + d(x_n, x_{n+1})} \bigg\}. \end{aligned} \tag{12}$$

Consider the subsets

$$N_1 = \{\, n \in \mathbb{N} : \text{ the maximum in (12) is } \ d(x_n, x_{n+1}) \,\},$$

$$N_2 = \left\{ n \in \mathbb{N} : \text{ the maximum in (12) is } \frac{d(x_n, x_{n+1})\,(1 + d(x_{n+1}, x_{n+2}))}{1 + d(x_n, x_{n+1})} \right\},$$

$$N_3 = \{ n \in \mathbb{N} : \text{ the maximum in (12) is } d(x_{n+1}, x_{n+2}) \}.$$

Clearly $N_1 \cup N_2 \cup N_3 = \mathbb{N}$. We claim that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

We can consider the following cases.

- If $n \in N_1$, then $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$, so (13) holds.
- If $n \in N_2$, then

  $$d\,(x_{n+1}, x_{n+2}) < \frac{d(x_n, x_{n+1})\,(1 + d(x_{n+1}, x_{n+2}))}{1 + d(x_n, x_{n+1})}.$$

  Hence

  $$\begin{aligned} d\,(x_{n+1}, x_{n+2}) &+ d(x_n, x_{n+1})\, d\,(x_{n+1}, x_{n+2}) \\ &< d(x_n, x_{n+1}) + d(x_n, x_{n+1})\, d(x_{n+1}, x_{n+2}), \end{aligned}$$

  which yields $d\,(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$, that is, (13) holds.
- If $n \in N_3$, then $d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x_{n+2})$, which is impossible.

In any case, we proved that (13) holds. Since $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of non-negative real numbers, it is convergent. Let $a \geq 0$ be its limit. Letting $n \to \infty$ in (11),

$$\lim_{n\to\infty} N\,(x_n, x_{n+1}) = \max \left\{ a, \frac{a\,(1 + a)}{1 + a}, a \right\} = a.$$

By (10), we have that, for all $n \in \mathbb{N}$,

$$0 \leq \psi\,(N(x_n, x_{n+1})) \leq \varphi(N(x_n, x_{n+1})) - \varphi\,(d\,(x_{n+1}, x_{n+2}))\,.$$

As φ is continuous, we deduce that

$$\lim_{n\to\infty} \psi\,(N(x_n, x_{n+1})) = 0,$$

and, as ψ is lower semi-continuous,

$$0 \leq \psi\,(a) \leq \liminf_{t\to a} \psi\,(t) \leq \lim_{n\to\infty} \psi\,(N(x_n, x_{n+1})) = 0.$$

Hence $\psi\,(a) = 0$ and $a = 0$, that is, $\{d(x_n, x_{n+1})\} \to 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in (X, d). Reasoning by contradiction, in the contrary case, it is well known (see, for instance, [**7, 14**]) that there exist $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,

$$k \leq n(k) < m(k), \quad d(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon_0 < d(x_{n(k)}, x_{m(k)})$$

and also

$$\begin{aligned}(14) \qquad \lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) &= \lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)}) \\ &= \lim_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \varepsilon_0.\end{aligned}$$

Applying the contractivity condition (8), it follows that, for all $k \in \mathbb{N}$,

$$\begin{aligned}(15) \qquad \varphi\left(d(x_{n(k)+1}, x_{m(k)+1})\right) &= \varphi\left(d(Tx_{n(k)}, Tx_{m(k)})\right) \\ &\leq \varphi\left(N(x_{n(k)}, x_{m(k)})\right) - \psi\left(N(x_{n(k)}, x_{m(k)})\right),\end{aligned}$$

where

$$\begin{aligned}&N(x_{n(k)}, x_{m(k)}) \\ &\quad = \max\left\{d(x_{n(k)}, x_{m(k)}), \frac{d(x_{n(k)}, Tx_{n(k)})(1 + d(x_{m(k)}, Tx_{m(k)}))}{1 + d(x_{n(k)}, x_{m(k)})}, \right. \\ &\qquad\qquad \left. \frac{d(x_{m(k)}, Tx_{m(k)})(1 + d(x_{n(k)}, Tx_{n(k)}))}{1 + d(x_{n(k)}, x_{m(k)})}\right\} \\ &\quad = \max\left\{d(x_{n(k)}, x_{m(k)}), \frac{d(x_{n(k)}, x_{n(k)+1})(1 + d(x_{m(k)}, x_{m(k)+1}))}{1 + d(x_{n(k)}, x_{m(k)})}, \right. \\ &\qquad\qquad \left. \frac{d(x_{m(k)}, x_{m(k)+1})(1 + d(x_{n(k)}, x_{n(k)+1}))}{1 + d(x_{n(k)}, x_{m(k)})}\right\}.\end{aligned}$$

Using (14) and letting $n \to \infty$ in the previous equality, we deduce that

$$(16) \qquad \lim_{k\to\infty} N(x_{n(k)}, x_{m(k)}) = \max\{\varepsilon_0, 0, 0\} = \varepsilon_0.$$

It follows from (15) that, for all $k \in \mathbb{N}$,

$$0 \leq \psi\left(N(x_{n(k)}, x_{m(k)})\right) \leq \varphi\left(N(x_{n(k)}, x_{m(k)})\right) - \varphi\left(d(x_{n(k)+1}, x_{m(k)+1})\right).$$

As φ is continuous, (14) and (16) leads to

$$\lim_{k\to\infty} \psi\left(N(x_{n(k)}, x_{m(k)})\right) = 0,$$

and, as ψ is lower semi-continuous,

$$0 \leq \psi(\varepsilon_0) \leq \liminf_{t\to\varepsilon_0} \psi(t) \leq \lim_{k\to\infty} \psi\left(N(x_{n(k)}, x_{m(k)})\right) = 0.$$

Hence $\psi(\varepsilon_0) = 0$ and $\varepsilon_0 = 0$, which contradicts the fact that $\varepsilon_0 > 0$. As a consequence, $\{x_n\}$ is a Cauchy sequence in (X, d). Since it is complete, then there exists $u \in X$ such that $\{x_n\} \to u$. We claim that u is a fixed point of T. Indeed, by using the contractivity condition (8), it follows that, for all $n \in \mathbb{N}$,

$$\varphi\left(d(x_{n+1}, Tu)\right) = \varphi\left(d(Tx_n, Tu)\right) \leq \varphi\left(N(x_n, u)\right) - \psi\left(N(x_n, u)\right), \tag{17}$$

where

$$N(x_n, u) = \max\left\{d(x_n, u), \frac{d(x_n, Tx_n)\ (1 + d(u, Tu))}{1 + d(x_n, u)}, \frac{d(u, Tu)\,(1 + d(x_n, Tx_n))}{1 + d(x_n, u)}\right\}$$

$$= \max\left\{\ d(x_n, u),\ \frac{d(x_n, x_{n+1})\ (1 + d(u, Tu))}{1 + d(x_n, u)}, \frac{d(u, Tu)\ (1 + d(x_n, x_{n+1}))}{1 + d(x_n, u)}\ \right\}.$$

Since $\{x_n\} \to u$, letting $n \to \infty$ in the previous inequality, we deduce that

$$\lim_{n\to\infty} N(x_n, u) = \max\left\{0, \frac{0(1 + d(u, Tu))}{1 + 0}, \frac{d(u, Tu)(1 + 0)}{1 + 0}\right\} = d(u, Tu).$$

By (17), for all $n \in \mathbb{N}$, we have that

$$0 \leq \psi\left(N(x_n, u)\right) \leq \varphi\left(N(x_n, u)\right) - \varphi\left(d(x_{n+1}, Tu)\right),$$

so, taking into account that φ is continuous,

$$\lim_{n\to\infty} \psi\left(N(x_n, u)\right) = 0.$$

As ψ is lower semi-continuous,

$$0 \leq \psi\left(d(u, Tu)\right) \leq \liminf_{t\to d(u,Tu)} \psi\left(t\right) \leq \lim_{n\to\infty} \psi\left(N(x_n, u)\right) = 0.$$

Hence $\psi\left(d(u, Tu)\right) = 0$ and $d(u, Tu) = 0$, which means that u is a fixed point of T.

Next, we prove that u is the unique fixed point of T. Let x and y be arbitrary fixed points of T. Using the contractivity condition (8), it follows that

$$\varphi\left(d(x, y)\right) = \varphi\left(d(Tx, Ty)\right) \leq \varphi\left(N(x, y)\right) - \psi\left(N(x, y)\right), \tag{18}$$

where

$$N(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)(1 + d(y, Ty))}{1 + d(x, y)}, \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)}\right\}$$

$$= \max\left\{d(x, y), \frac{0(1 + 0)}{1 + d(x, y)}, \frac{0(1 + 0)}{1 + d(x, y)}\right\} = d(x, y).$$

Then (18) becomes

$$\varphi\left(d(x, y)\right) \leq \varphi\left(d(x, y)\right) - \psi\left(d(x, y)\right).$$

If we suppose that $x \neq y$, then $d(x, y) > 0$, and (9) guarantees that $d(x, y) < d(x, y)$, which is impossible. Then $x = y$ and T has a unique fixed point. □

In the context of ordered metric spaces, we can prove the following analog of Theorem 7.

Theorem 17. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that there exist $L \geq 0$ and a continuous function $\phi \in \mathcal{F}_{\mathrm{com}}$ verifying*

$$d(Tx, Ty) \leq \phi\bigg(\max\Big\{ d(x,y), \frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)}, \frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)} \Big\} \bigg) + L \min\big\{ d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx) \big\}$$

for all $x, y \in X$ with $x \preccurlyeq y$. Assume that, at least, one of the following conditions holds:

(*a*) *T is continuous, or*

(*b*) *If $\{x_n\} \subseteq X$ is a nondecreasing sequence such that $\{x_n\} \to u \in X$, then $x_n \preccurlyeq u$ for all $n \in \mathbb{N}$.*

If there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, then T has a fixed point.

Proof. Starting from the point $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, let define $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, so x_{n_0} is a fixed point of T. In this case, the proof is complete. Suppose, on the contrary, that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, that is, $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. As $x_0 \preccurlyeq Tx_0 = x_1$ and T is non-decreasing, then $x_1 = Tx_0 \preccurlyeq Tx_1 = x_2$. By induction, we deduce that $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$. And as $\preccurlyeq$ is transitive, we have that $x_n \preccurlyeq x_m$ for all $n, m \in \mathbb{N}$ such that $n \leq m$. Following, point by point, the arguments of the proof of Theorem 7, we may deduce that $\{x_n\}$ is a Cauchy sequence. As (X, d) is complete, there exists $u \in X$ such that $\{x_n\} \to u$. In order to prove that u is a fixed point of T, we can consider the following two cases.

- If T is continuous, it follows that $\{x_{n+1} = Tx_n\} \to Tu$, and the uniqueness of the limit of a convergent sequence in a metric space guarantees that $Tu = u$.
- If assumption (*b*) holds, then $x_n \preccurlyeq u$ for all $n \in \mathbb{N}$, so the contractivity condition is applicable. As a consequence, following, step by step, the arguments of the proof of Theorem 7, we conclude that u is a fixed point of T.

Either way, T has, at least, a fixed point.

4. AN APPLICATION TO DYNAMIC PROGRAMMING

In order to illustrate our results, we present the study about the existence and uniqueness of solutions of the functional equation

$$u(x) = \sup_{y \in D} \big\{ g(x,y) + F(x,y,u(T(x,y))) \big\} \tag{19}$$

which appears in dynamic programming (see [**3**]), where $x \in S$, and S is a state space, D is a decision space, $T : S \times D \to S$, $g : S \times D \to \mathbb{R}$ and $F : S \times D \times \mathbb{R} \to \mathbb{R}$ are given mappings. First, we need the following lemma.

Lemma 18. *If $G, H : S \to \mathbb{R}$ are two bounded functions, then*

$$\Big| \sup_{x \in S} G(x) - \sup_{x \in S} H(x) \Big| \leq \sup_{x \in S} \left| G(x) - H(x) \right| .$$

We consider a nonempty set S and we work in the space $\mathrm{B}(S)$ which denotes the family of all bounded real functions defined on S. According to the ordinary addition of functions and scalar multiplication, and with the norm $\|\cdot\|_\infty$ given by

$$\|u\|_\infty = \sup_{x \in S} \left| u(x) \right| \quad \text{for all } u \in \mathrm{B}(S),$$

we have that $(\mathrm{B}(S), \|\cdot\|_\infty)$ is a Banach space. In fact, the distance in $\mathrm{B}(S)$ is given by

$$d_\infty(u, v) = \sup_{x \in S} \left| u(x) - v(x) \right| \quad \text{for all } u, v \in \mathrm{B}(S).$$

Lemma 19. *Suppose the following assumptions:*

(i) *$g : S \times D \to \mathbb{R}$ and $F(\cdot, \cdot, 0) : S \times D \to \mathbb{R}$ are bounded functions.*

(ii) *There exists $M \geq 0$ such that, for all $x \in S$, $y \in D$ and $t, r \in \mathbb{R}$,*

$$\big| F(x, y, t) - F(x, y, r) \big| \leq M \, \left| t - r \right| .$$

Then the operator $R : \mathrm{B}(S) \to \mathrm{B}(S)$ given, for all $u \in \mathrm{B}(S)$ and all $x \in S$, by

$$(Ru)(x) = \sup_{y \in D} \big\{ g(x, y) + F(x, y, u(T(x, y))) \big\}, \tag{20}$$

is well defined.

Proof. We only need to prove that, for all $u \in \mathrm{B}(S)$, the function $Ru : S \to \mathbb{R}$ is bounded. Indeed, let $u \in \mathrm{B}(S)$ be arbitrary. As u is bounded, there exists $M_1 > 0$ such that

$$\left| u(x) \right| \leq M_1 \quad \text{for all } x \in S.$$

By hypothesis (i), there exist $M_2, M_3 > 0$ such that, for all $x \in S$ and all $y \in D$,

$$\left| g(x, y) \right| \leq M_2 \quad \text{and} \quad \left| F(x, y, 0) \right| \leq M_3.$$

Notice that, for all $x \in S$ and all $y \in D$,

$$\begin{aligned} \left| g(x, y) + F(x, y, u(T(x, y)) \right| &\leq \left| g(x, y) \right| + \left| F(x, y, u(T(x, y)) \right| \\ &\leq M_2 + \left| F(x, y, u(T(x, y)) - F(x, y, 0) \right| + \left| F(x, y, 0) \right| \\ &\leq M_2 + M \, \left| u(T(x, y) \right| + M_3 \leq M_2 + M \cdot M_1 + M_3. \end{aligned}$$

As a consequence, for all $x \in S$, we have that

$$|\,(Ru)(x)\,| \leq \sup_{y\in D} |\,g(x,y)+F(x,y,u(T(x,y))\,| \leq M_2 + M\cdot M_1 + M_3,$$

which means that Ru is a bounded function on S, that is, $Ru \in \mathrm{B}(S)$ and the operator R is well defined. □

Now, we are ready to present the following result.

Theorem 20. *Suppose the following assumptions:*

(i) $g: S\times D\to\mathbb{R}$ *and* $F(\cdot,\cdot,0): S\times D\to\mathbb{R}$ *are bounded functions.*

(ii) *There exists* $M\geq 0$ *such that, for all* $x\in S$, $y\in D$ *and* $t,r\in\mathbb{R}$,

$$\left|F(x,y,t)-F(x,y,r)\right| \leq M\ |\,t-r\,|\,.$$

(iii) *There exists a continuous comparison function* $\phi\in\mathcal{F}_{\mathrm{com}}$ *such that, for all* $x\in S$, *all* $y\in D$ *and all* $u,v\in\mathrm{B}(S)$,

$$\begin{aligned}&\left|F(x,y,u(T(x,y))\ -F(x,y,v(T(x,y))\right|\\ &\leq \phi\left(\max\left\{d_\infty(u,v),\frac{d_\infty(u,Ru)(1+d_\infty(v,Rv))}{1+d_\infty(u,v)},\frac{d_\infty(v,Rv)(1+d_\infty(u,Rx))}{1+d_\infty(u,v)}\right\}\right)\\ &\qquad + L\min\left\{d_\infty(u,Ru),d_\infty(v,Rv),d_\infty(u,Rv),d_\infty(v,Ru)\right\}.\end{aligned}$$

Then Problem (19) *has a unique solution* $u_0\in\mathrm{B}(S)$.

Proof. Consider the operator $R:\mathrm{B}(S)\to\mathrm{B}(S)$ given in (20), which is well defined by Lemma 19. Next, we check that R satisfies the contractivity condition (1). Indeed, using Lemma 18 and the non-decreasing character of ϕ, we deduce that, for all $u,v\in\mathrm{B}(S)$ and all $x\in S$,

$$\begin{aligned}&\left|(Ru)(x)-(Rv)(x)\right|\\ &\quad=\left|\sup_{y\in D}\{\,g(x,y)+F(x,y,u(T(x,y))\,\}-\sup_{y\in D}\{\,g(x,y)+F(x,y,v(T(x,y)))\}\,\right|\\ &\quad\leq\sup_{y\in D}\left|\,(\,g(x,y)+F(x,y,u(T(x,y))\,)-(\,g(x,y)+F(x,y,v(T(x,y))))\right|\\ &\quad=\sup_{y\in D}|\,F(x,y,u(T(x,y))\ -F(x,y,v(T(x,y))\,|\\ &\quad\leq\phi\left(\max\left\{d_\infty(u,v),\frac{d_\infty(u,Ru)(1+d_\infty(v,Rv))}{1+d_\infty(u,v)},\frac{d_\infty(v,Rv)(1+d_\infty(u,Rx))}{1+d_\infty(u,v)}\right\}\right)\\ &\qquad\quad+L\min\left\{d_\infty(u,Ru),d_\infty(v,Rv),d_\infty(u,Rv),d_\infty(v,Ru)\right\}.\end{aligned}$$

As a consequence,

$$d_\infty(Ru,Rv)=\sup_{x\in S}\left|(Ru)(x)-(Rv)(x)\right|$$

$$\leq \phi\Big(\max\Big\{ d_\infty(u,v), \frac{d_\infty(u,Ru)(1+d_\infty(v,Rv))}{1+d_\infty(u,v)} \frac{d_\infty(v,Rv)\ (1+d_\infty(u,Rx))}{1+d_\infty(u,v)} \Big\}\Big)$$
$$+ L\min\{d_\infty(u,Ru), d_\infty(v,Rv), d_\infty(u,Rv), d_\infty(v,Ru)\ \}$$

for all $u, v \in \mathrm{B}(S)$, which means that R satisfies all hypotheses of Theorem 7. Thus, there exists a unique $u_0 \in \mathrm{B}(S)$ such that $Ru_0 = u_0$. Hence, for all $x \in S$,

$$u_0(x) = (Ru_0)(x) = \sup_{y\in D}\ \{ g(x,y) + F(x,y,u_0(T(x,y)) \} .$$

This completes the proof.

Corollary 21. *Suppose the following assumptions*:

(i) $g : S \times D \to \mathbb{R}$ *and* $F(\cdot,\cdot,0) : S\times D \to \mathbb{R}$ *are bounded functions.*

(ii) *There exists a continuous comparison function* $\phi \in \mathcal{F}_{\mathrm{com}}$ *such that, for all* $x\in S$, $y \in D$ *and* $t, r \in \mathbb{R}$,

$$|\, F(x,y,t) - F(x,y,r)\,| \leq \phi\,(|\, t-r\,|)\,.$$

Then Problem (19) *has a unique solution* $u_0 \in \mathrm{B}(S)$.

Proof. We claim that Theorem 20 is applicable. Since ϕ verifies axiom $(\mathcal{P}_3)$, then we can take $M = 1$ in item (ii) of Theorem 20. Moreover, for all $x \in S$, all $y \in D$ and all $u, v \in \mathrm{B}(S)$,

$$|\, u(T(x,y)) - v(T(x,y))\,| \leq \sup_{x\in S}\ |\, u(x) - v(x)\,| = d_\infty(u,v).$$

As ϕ is nondecreasing,

$$\big|F(x,y,u(T(x,y)))\, F(x,y,v(T(x,y)))\big|$$
$$\leq \phi\big(\,|\, u(T(x,y)) - v(T(x,y))\,|\,\big) \leq \phi\,(d_\infty(u,v))$$
$$\leq \phi\Big(\max\Big\{d_\infty(u,v), \frac{d_\infty(u,Ru)(1+d_\infty(v,Rv))}{1+d_\infty(u,v)}, \frac{d_\infty(v,Rv)(1+d_\infty(u,Rx))}{1+d_\infty(u,v)}\Big\}\Big)$$
$$+ L\min\{d_\infty(u,Ru), d_\infty(v,Rv), d_\infty(u,Rv), d_\infty(v,Ru)\ \}\,.$$

Hence, it follows that Theorem 20 guarantees that Problem (19) has a unique solution $u_0 \in \mathrm{B}(S)$. □

Next, we present the following numerical example.

Example 22. Let consider the following functional equation:

$$u(x) = \sup_{y\in\mathbb{R}} \Big\{ \arctan\,(x + 3\ |y|) + \ln\Big(1 + x + \frac{1}{1+|y|} + \Big|u\Big(\frac{x}{1+x+|y|}\Big)\Big|\Big)\Big\} \tag{21}$$

for $x \in [0, 1]$. Notice that Equation (21) is a particular case of Equation (19) where

$$S = [0,1]\,, \quad D = \mathbb{R},$$

$$g : [0,1] \times \mathbb{R} \to \mathbb{R} \quad \text{is defined by} \quad g(x,y) = \arctan\left(x + 3\,|y|\right),$$
$$T : [0,1] \times \mathbb{R} \to \mathbb{R} \quad \text{is defined by} \quad T(x,y) = \frac{x}{1+x+|y|} \quad \text{and}$$
$$F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \text{is defined by} \quad F(x,y,t) = \ln\left(1 + x + \frac{1}{1+|y|} + |\,t\,|\right).$$

It is clear that $|\,g(x,y)\,| \leq \pi/2$ and

$$|\,F(x,y,0)\,| = \left|\ln\left(1+x+\frac{1}{1+|y|}\right)\right| \leq \ln 3$$

for all $x \in [0,1]$ and all $y \in \mathbb{R}$. Hence, assumption (i) of Theorem 20 is satisfied. Furthermore, consider the continuous comparison function φ given by $\varphi(t) = \ln(1+t)$ for all $t \in [0,\infty)$. Therefore, for all $x \in [0,1]$ and all $y,t,r \in \mathbb{R}$ (we can assume that $|\,t\,| > |\,r\,|$ without loss of generality), it follows that

$$\begin{aligned}
&|\,F(x,y,t) - F(x,y,r)\,| \\
&\quad = \left|\ln\left(1+x+\frac{1}{1+|y|}+|t|\right) - \ln\left(1+x+\frac{1}{1+|y|}+|r|\right)\right| \\
&\quad = \left|\ln\frac{1+x+\frac{1}{1+|y|}+|t|}{1+x+\frac{1}{1+|y|}+|r|}\right| = \left|\ln\frac{1+x+\frac{1}{1+|y|}+|\,r\,|+(|t|-|\,r\,|)}{1+x+\frac{1}{1+|y|}+|\,r\,|}\right| \\
&\quad = \left|\ln\left(1+\frac{|t|-|r|}{1+x+\frac{1}{1+|y|}+|r|}\right)\right| \leq |\ln\left(1+(|\,t\,|-|\,r\,|)\right)| \\
&\quad = \ln\left(1+(|\,t\,|-|\,r\,|)\right) = \ln\left(1+\big|\,|\,t\,|-|\,r\,|\,\big|\right) \\
&\quad \leq \ln\left(1+|\,t-r\,|\right) = \varphi\left(|\,t-r\,|\right).
\end{aligned}$$

As a result, assumption (ii) of Corollary 21 is also satisfied, which implies that Problem (21) has a unique solution $u_0 \in \mathrm{B}([0,1])$.

Example 23. Assume that $S = \{s_0\}$ is a singleton. Then $(\mathrm{B}(S), \|\cdot\|_\infty)$ is isometric to $(\mathbb{R}, |\cdot|)$ because each function $u : \{s_0\} \to \mathbb{R}$ can be identified with its image (which is a unique real number). Let D be an arbitrary nonempty set and let consider the functions $g : \{s_0\} \times D \to \mathbb{R}$ and $F : \{s_0\} \times D \times \mathbb{R} \to \mathbb{R}$:

$$g(s_0,y) = 0, \quad F(s_0,y,r) = \begin{cases} r/2, & \text{if } r \in [0,2]\,, \\ 3-r, & \text{if } r \in (2,3]\,, \\ 0, & \text{otherwise,} \end{cases}$$

for all $y \in D$ and all $r \in \mathbb{R}$. In this case, for all $u \in \mathrm{B}(S) \equiv \mathbb{R}$,

$$\begin{aligned}
(Ru)(s_0) &= \sup_{y\in D}\,\{\,g(s_0,y) + F(s_0,y,u(T(x,y)))\,\} \\
&= \begin{cases} u/2, & \text{if } u \in [0,2]\,, \\ 3-u, & \text{if } u \in (2,3]\,, \\ 0, & \text{otherwise,} \end{cases}
\end{aligned} \tag{22}$$

This operator can be seen as $R : \mathbb{R} \to \mathbb{R}$ given by (22), which was already studied in Example 11. In particular, we proved that R satisfies the contractivity condition (1) using $\phi_{1/2} \in \mathcal{F}_{\mathrm{com}}$ and $L = 0$, where $\phi_{1/2}(t) = t/2$ for all $t \in [0, \infty)$. However, as we pointed out in Remark 12, we cannot apply to R neither Boyd and Wong's theorem nor Matkowski's theorem. However, Theorem 7 guarantees that R has a unique fixed point (which is $u \equiv 0$).

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REFERENCES

1. R. Agarwal, E. Karapinar, A.F. Roldán-López-de-Hierro: *Fixed point theorems in quasi-metric spaces and applications to coupled/tripled fixed points on G^*-metric spaces.* J. Nonlinear Convex Anal., (in press).
2. A. Aghajani, J. Banas, N. Sabzadi: *Some generalizations of Darbo fixed point theorem and applications.* Bull. Belg. Math. Soc. Simon Stevin, **20** (2013), 345–358.
3. R. Bellman, E. S. Lee: *Functional equations in dynamic programming.* Aequationes Math., **17** (1978), 1–18.
4. V. Berinde: *Iterative Approximation of Fixed Points.* Editura Efemeride, Baia Mare, 2002.
5. V. Berinde: *Contracţii generalizate şi aplicaţii.* Editura Cub Press 22, Baia Mare, Romania, 1997.
6. V. Berinde: *A common fixed point theorem for compatible quasi contractive self mappings in metric spaces.* Appl. Math. Comput., **213** (2009), 348–354.
7. M. Berzig, E. Karapinar, A.-F. Roldán-López-de-Hierro: *Discussion on generalized-$(\alpha\psi, \beta\varphi)$-contractive mappings via generalized altering distance function and related fixed point theorems.* Abstr. Appl. Anal. 2014, Article ID 259768, 12 pages.
8. D. W. Boyd, J. S. W. Wong: *On nonlinear contractions.* Proc. Amer. Math. Soc., **20** (1969), 458–464.
9. I. Cabrera, J. Harjani, K. Sadarangani: *A fixed point theorem for contractions of rational type in partially ordered metric spaces.* Ann. Univ. Ferrara Sez. VII Sci. Mat., **59** (2013), 251–258.
10. N. V. Can, N. X. Thuan: *Fixed point theorem for generalized weak contractions involving rational expressions.* Open Journal of Mathematical Modelling, **1** (2013), 29–33.
11. B. K. Dass, S. Gupta: *An extension of Banach contraction principle through rational expressions.* Indian J. Pure Appl. Math., **6** (1975), 1455–1458.

12. J. Jachymski: *Equivalent conditions for generalized contractions on (ordered) metric spaces.* Nonlinear Anal., **74** (2011), 768–774.

13. J. Matkowski: *Fixed point theorems for mappings with a contractive iterate at a point.* Proc. Amer. Math. Soc., **62** (1977), 344–348.

14. A. Roldán, J. Martínez-Moreno, C. Roldán, E. Karapinar: *Multidimensional fixed-point theorems in partially ordered complete partial metric spaces under (ψ, φ)-contractivity conditions.* Abstr. Appl. Anal. 2013, Article ID 634371, 12 pages.

15. A. F. Roldán-López-de-Hierro, N. Shahzad: *Some fixed/coincidence point theorems under (ψ, φ)-contractivity conditions without an underlying metric structure.* Fixed Point Theory and Applications 2014, 2014:218, 24 pages.

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