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DYNAMICS OF A NONLINEAR DISCRETE POPULATION MODEL WITH JUMPS

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Our aim is to investigate the global asymptotic behavior, the existence of invariant intervals, oscillatory behavior, structure of semicycles, and periodicity of a nonlinear discrete population model of the form $x_{n+1}= F(x_n)$, for $n = 0, 1, \ldots,$where $x_0> 0$, and the function F is a positive piecewise continuous function with two jump discontinuities satisfying some additional conditions. The motivation for study of this general model was inspired by the classical Williamson's discontinuous population model, some recent results about the dynamics of the discontinuous Beverton-Holt model, and applications of discontinuous maps to the West Nile epidemic model.
In the first section we introduce the population model which is a focal point of this paper. We provide background information including a summary of related results, a comparison between characteristics of continuous and discontinuous population models (with and without the Allee-type effect), and a justification of hypotheses introduced in the model. In addition we review some basic concepts and formulate known results which will be used later in the paper. The second and third sections are dedicated to the study of the dynamics and the qualitative analysis of solutions of the model in two distinct cases. An example, illustrating the obtained results, together with some computer experiments that provide deeper insight into the dynamics of the model are presented in the fourth section. Finally, in the last section we formulate three open problems and provide some concluding remarks.

1. INTRODUCTION

In recent years there has been increased interest in studying the dynamics of discontinuous maps. The simplest discontinuous map is the one-dimensional

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piecewise linear map

$$x \longmapsto \begin{cases} \nu_1 x + \mu, & \text{if } x \leq 0, \\ \nu_2 x + \mu + \lambda, & \text{if } x > 0, \end{cases} \tag{1}$$

where $\nu_1, \nu_2, \mu, \lambda \in \mathbb{R}$. Such a simple map possesses complex and rich dynamics. The summary of basic characteristics of (1) can be found in [**12**] and [**24**]. In several papers [**6, 7, 8, 10, 12, 14, 19, 20, 22, 23, 24**] the study of the dynamics was directed toward a detailed understanding of bifurcations and the structure of periodic solutions. In continuous systems, the transition from regular to complex dynamics (such as chaos) occurs through a sequence of bifurcations (for example, period-doubling). In discontinuous systems, in contrast, more prevalent are non-standard bifurcations such as a border-collision bifurcation with a sharp transition from regular dynamics to chaos. Other exotic types of bifurcations such as period adding bifurcations, period incriminating bifurcations, and Farey tree type bifurcations are very common in this equation.

The discontinuous maps found applications in many different areas such as neural networks [**28, 29, 41, 42**], flip-flop processes in the Lorenz flow [**38, 39**], and economics [**4**].

The applications of discontinuous systems in mathematical biology are very scarce. The following population model was originally introduced by Williamson [**37**] (involving the age structure as well), and simplified by May and Oster [**30**] (see also [**31, 32**]):

$$N_{n+1} = \begin{cases} \lambda^+ N_n, & \text{if } N_n \leq K, \\ \lambda^- N_n, & \text{if } N_n > K, \end{cases} \tag{2}$$

where $0 < \lambda^- < 1 < \lambda^+$, $K > 0$, and $N_n > 0$ represents population size in generation n. Felsenstein [**18**] analyzed some properties of model (2). Kocic [**25**] investigated the oscillatory behavior, gave the detailed description of semicycles and obtained some results about the existence of periodic orbits for the piecewise linear equation of the form

$$x_{n+1} = (a - bh(x_n - c))x_n, \tag{3}$$

where $x_0 \geq 0$ and a, b, and c are positive constants such that $0 < b < 1 < a < b+1$ and h is the Heaviside function $h(t) = 0$ for $t < 0$ and $h(t) = 1$ for $t \geq 0$. The motivation for studying the dynamics of equation (3) came from the discrete model of the West Nile-like epidemics (see [**25**] and references cited therein). Note that equation (3) becomes Williamson's population model (2) when $\lambda^- = a, \lambda^+ = a - b$, and $K = c$. Recently, in [**26**], the Beverton-Holt model with two discontinuities,

$$x_{n+1} = \frac{k(x_n) r(x_n) x_n}{k(x_n) + (r(x_n) - 1)x_n}, \quad n = 0, 1, \ldots, \tag{4}$$

where $x_0 > 0$ and $r(x) = R + Sh(x - T)$, $k(x) = K + Lh(x - M)$, with $L, S \in \mathbb{R}$, $K > \max\{0, -L\}$, $R > \max\{1, -S+1\}$, $T, M > 0$, and h is the Heaviside function, was introduced and studied. Depending on the values of parameters the model may have two equilibria, one equilibrium, or no equilibria. Other discontinuous maps with two discontinuities, including piecewise linear equations and some nonlinear equations with quadratic nonlinearities, can be found in [**34, 35, 36**].

According to Cull [**17**], the discrete population model (without delay and not exhibiting the Allee effect) is a difference equation

$$x_{n+1} = \varphi(x_n), \quad n = 0, 1, \dots$$

where $\varphi \in C[[0,\infty),[0,\infty)]$ and there exists a positive equilibrium $\bar{x}$ such that

$$\begin{aligned} &\varphi(0) = 0, \quad \varphi(x) > x \quad \text{for } x \in (0,\bar{x}), \\ &\varphi(x) = x \quad \text{for } x = \bar{x}, \quad \varphi(x) < x \quad \text{for } x \in (\bar{x},\infty) \end{aligned}$$

and if $\varphi'(x_m) = 0$ and $x_m \le \bar{x}$, then

$$\varphi'(x) > 0 \quad \text{for } x \in [0,\bar{x}), \qquad \varphi'(x) < 0 \quad \text{for } x \in (\bar{x},\infty) \text{ such that } \varphi(x) > 0.$$

The concept of permanence plays a very important role in population dynamics. A difference equation $x_{n+1} = f(x_n)$, where $f : [0,\infty) \to [0,\infty)$, is said to be **permanent** (see, for example, [**3, 27**]) if there exist numbers $0 < C \le D < \infty$ such that for any initial condition $x_0 \in (0,\infty)$ there exists a positive integer N which depends on initial conditions such that

$$C \le x_n \le D \qquad \text{for} \qquad n \ge N.$$

Anderson, Hutson, and Law [**3**] considered the role of permanence in population dynamics to be superior to that of asymptotic stability: "The limitations of asymptotic stability are well-known; the analysis tells us only what happens to the community dynamics in the immediate neighborhood of the equilibrium under investigation. An unstable equilibrium would not necessarily imply that one or more species will be driven to extinction, because all of the species might still coexist on cyclical and chaotic orbits." Also they mentioned [**3**]: "In this note we consider another alternative, that of permanence, which we believe to be superior to that of asymptotic stability and which is relatively tractable."

Clearly, Williamson's population model (2) does not satisfty the hypotheses of the population model according to Cull. In Williamson's population model, the function φ defined by

$$\varphi(x) = \begin{cases} \lambda^+ x, & \text{if } x \le K, \\ \lambda^- x, & \text{if } x > K, \end{cases}$$

($0 < \lambda^- < 1 < \lambda^+$, $K > 0$) is discontinuous at $x = K$ and does not have a positive equilibrium. However, all other hypotheses are satisfied under the assumption that the discontinuity K replace the equilibrium $\bar{x}$. Also, in [**25**] the existence of an invariant interval was established for equation (3) which is equivalent to

Williamson's model (2). This implied the existence of permanence in Williamson's model. In other words in discontinuous population models, it is natural to have discontinuities instead of equlibria and permanence instead of asymptotic stability.

In population dynamics, the "Allee effect" is broadly defined (see [**2**]) as a "... positive relationship between any component of individual fitness and either numbers or density of conspecifics." Practically speaking, the Allee effect causes, at low population densities, per capita birth rate decline. Under such a scenario, at low population densities, the population may slide into extinction. There are various scenarios in which the Allee effect appears in nature (see [**16**] and references cited therein) and can be attributed to different reasons. These include, for example, difficulties in finding mates when the population size is small, higher mortality rate in juveniles when there are not enough adults to protect them from predation, or uncontrollable harvesting, as in overfishing. On the other hand, the Allee effect can be beneficial in some situations such as in controlling a population of fruit flies, which are considered to be among the worst insect pests in agriculture [**16**]. The technique used to control fruit flies is the release of sterile males, which creates the Allee effect. Several discrete mathematical models exhibiting the Allee effect are known and have been reported in the literature (see [**13**] and the references cited therein). They all have the following common properties:

(i) existence of three equilibrium points: 0, T - Allee threshold, and K - carrying capacity of the environment $(0 < T < K)$;

(ii) equilibria 0 and K are stable, while T is unstable;

(iii) if the population size drops below T, then the population slides into extinction, so it approaches 0.

Our main goal in this paper is to initiate the study of the dynamics of nonlinear discontinuous population models. We will focus on the model

$$x_{n+1} = F(x_n), \quad n = 0, 1, \dots, \tag{5}$$

where $x_0 > 0$ and the function F satisfies the following hypotheses:

(H$_1$)

$$F(x) = \begin{cases} f(x), & \text{if } x \in [0, a) \\ g(x), & \text{if } x \in [a, b) \\ h(x), & \text{if } x \in [b, \infty) \end{cases} \tag{6}$$

such that $f \in C\big[[0,a],[0,\infty)\big]$, $g \in C\big[[a,b],(0,\infty)\big]$, and $h \in C\big[[b,\infty),(0,\infty)\big]$.

(H$_2$) The functions f, g, and h are increasing on their respective domains.

(H$_3$) $f(x) < x$ for $x \in (0, a]$, $g(x) > x$ for $x \in [a, b]$, and $h(x) < x$ for $x \in [b, \infty)$.

(H$_4$) $f(0) = 0$ and $\lim\limits_{x\to\infty} h(x) = H > 0$.

This is a generic model, exhibiting some properties of an Allee-type effect. Namely, the positive equilibria T and K in classical models with Allee effect are replaced with two discontinuities a and b in equation (5). This is in line with the way Williamson's model differs from classical population models. Our model, in particular the hypothesis (H_3) resembles some properties of typical continuous population model exhibiting Allee effect with two jump discontinuities in lieu of equilibria. Some variations in hypotheses may lead to discontinuous population models which may posses additional equilibria, in addition to discontinuities, similarly to discontinuous Beverton-Holt model (4) [**26**]. Such considerations are outside of the scope of this paper.

In order to accommodate characteristics of discontinuous models (which may or may not have equilibria), we introduce a weaker version of the Allee-type effect:

(i′) there exists at least one equilibrium point: 0;

(ii′) there exists a point $T > 0$ such that all solutions with initial conditions in $(0, T)$ are attracted to 0;

(iii′) all solutions with initial conditions in $[T, \infty)$ become trapped in an interval $[T, S]$, for some $S > T$.

In this paper in particular we will study the global asymptotic behavior, existence of invariant intervals (permanence), oscillation, and periodicity of a nonlinear discrete population model with jumps (5).

A sequence $\{x_n\}$ is said to **oscillate about zero** or simply to **oscillate** if the terms x_n are neither eventually all positive nor eventually all negative. Otherwise the sequence is called **nonoscillatory**. A sequence $\{x_n\}$ is said to **oscillate about** $\bar{x}$ if the sequence $\{x_n - \bar{x}\}$ oscillates. A **positive semicycle** of $\{x_n\}$ **with respect to** $\bar{x}$ is a "string" of terms $C_+ = \{x_{l+1},\ x_{l+2},\ \ldots, x_m\}$ such that $x_i \geq \bar{x}$, for $i = l+1, \ldots, m$, with $l \geq -1$ and $m \leq \infty$ and such that either $l = -1$ or $l \geq 0$ and $x_l < \bar{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} < \bar{x}$. A **negative semicycle** of $\{x_n\}$ **with respect to** $\bar{x}$ consists of a "string" of terms $C_- = \{x_{j+1},\ x_{j+2}, \ldots, x_l\}$, such that $x_i < \bar{x}$, for $i = j+1, \ldots, l$, with $j \geq -1$ and $l \leq \infty$ and such that either $j = -1$ or $j \geq 0$ and $x_j \geq \bar{x}$ and either $l = \infty$ or $l < \infty$ and $x_{l+1} \geq \bar{x}$. The first semicycle of a solution starts with the term x_0 and is positive if $x_0 \geq \bar{x}$ and negative if $x_0 < \bar{x}$. A solution may have a finite number of semicycles or infinitely many. The **length of a semicycle** is the number of terms in the semicycle.

Theorem A (Brouwer Fixed Point Theorem [**40**]). *The continuous operator $A : M \to M$ has at least one fixed point when M is a compact, convex, nonempty set in a finite dimensional normed space over* $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

The following result is the special case of more general result, popularly known as "M-m" Theorem (see for example [**15**] Theorem 1.6.5).

Theorem B. Let $f \in C([a, b], [a, b])$ be increasing and have the unique equilibrium $\bar{x} \in [a, b]$. Then $\bar{x}$ attracts all solutions of the difference equation $x_{n+1} = f(x_n)$, with initial condition $x_0 \in [a, b]$.

Section 2 is dedicated to the case when $h(b) > a$. Some of the results from this section are generalizations of similar results for the discontinuous Beverton-Holt model (4) studied recently in [**26**]. In Section 3 we focus on the case when $h(b) < a$. An example, illustrating results from Sections 2 and 3, is introduced in Section 4. In addition to theoretical results that follow from the general results, several computer simulations indicate very complex and interesting dynamics. Finally, in Section 5 we present some open problems and concluding remarks.

2. THE CASE $h(b) \geq a$

In this section we study the existence of equilibria, their attractivity, and some properties of solutions of equation (5) in the case when

$$h(b) \geq a. \tag{7}$$

The next technical lemma will be useful in the sequel.

Lemma 1. *Assume* $(H_1)-(H_4)$ *are satisfied and* $x_0 \in [0, a)$. *Then the corresponding solution* $\{x_n\}$ *is decreasing and converges to* 0.

The proof is trivial and it is omitted.

Lemma 2. *Assume* $(H_1)-(H_4)$ *and* (7) *are satisfied. Then the following inequalities hold*: $g(h(b)) > h(b)$ *and* $h(g(b)) < g(b)$.

Proof. Since $h(b) \geq a$ and $h(b) < b$, by using (H$_3$) we obtain $g(h(b)) > h(b)$. Similarly, since $g(b) > b$, by using (H$_3$) we obtain $h(g(b)) < g(b)$.

The next result establishes the existence of an invariant interval for f.

Theorem 3. *Assume* $(H_1)-(H_4)$ *and* (7) *are satisfied. Let* F *be defined by* (6) *Then the following statements are true*:

(i) *The interval* $I = [h(b), g(b)]$ *is invariant under* F, *that is,* $F([h(b), g(b)]) \subseteq [h(b), g(b)]$.

(ii) *If* $x_0 \in [0, a)$, *then* $\{x_n\}$ *is decreasing and converges to* 0.

(iii) *All positive solutions of equation* (5) *with initial conditions in* $[a, \infty)$ *become trapped in an interval* I.

(iv) *All solutions with initial conditions in* $[a, \infty)$ *strictly oscillate about* b.

(v) *No solution with initial condition in* $[a, \infty)$ *converges.*

Proof. (i) Using Lemma 2, for $x \in [h(b), b) \subset [a, b)$, we obtain

$$F(x) = g(x) < g(b),$$
$$F(x) = g(x) \geq g(h(b)) > h(b).$$

Similarly, for $x \in [b, g(b)]$ we find

$$F(x) = h(x) \geq h(b),$$
$$F(x) = h(x) \leq h(g(b)) < g(b),$$

which implies $F([h(b), g(b)]) \subseteq [h(b), g(b)]$, and the proof of part (i) is complete.
(ii) Let $x_0 \in (0, a)$. Then $x_1 = F(x_0) = f(x_0) < x_0$, so by induction the sequence $\{x_n\}$ is decreasing, and since it is bounded, it converges to 0.
(iii) Observe that for $x \in [a, \infty)$, $F(x) \geq \min_{x \in [a,\infty)} F(x) = \min\{g(a), h(b)\} \geq a$, so $F([a, \infty)) \subset [a, \infty)$. Assume, for the sake of contradiction, that $\{x_n\}$ is a positive solution of equation (5) which is not trapped in the invariant interval $I = [h(b), g(b)]$. Then, for every n, $x_n \notin I$. We will first consider the case when $h(b) > a$. Let $\{x'_j\}$ and $\{x''_j\}$ be subsequences of $\{x_n\}$ such that $a \leq x'_j < h(b)$ and $x''_j > g(b)$. Clearly, at most one of the subsequences $\{x'_j\}$ and $\{x''_j\}$ may have only a finite number of terms or no terms at all. Assume that $\{x''_j\}$ has a finite number of terms. Then, for sufficiently large n, $x_n < h(b) < b$. Furthermore $x_{n+1} = F(x_n) = g(x_n) > a$, so the sequence $\{x_n\}$ is eventually increasing and convergent. Let $x = \lim\limits_{n\to\infty} x_n \in [a, h(b)]$. Then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$$

which contradicts (H_3). The similar conclusion follows when $\{x'_j\}$ has only a finite number of terms. So the remaining case is when both $\{x'_j\}$ and $\{x''_j\}$ have infinitely many terms. Let, for some $m \in \mathbb{N}$, $x_m < h(b) < b$ and $x_{m+1} > g(b)$. Then

$$g(b) < x_{m+1} = F(x_m) = g(x_m) < g(b).$$

This is a contradiction. Finally, in the case when $h(b) = a$, all terms of the sequence $\{x_n\}$ satisfy $x_n > g(b)$, which is identical to the case (addressed above) when the subsequence $\{x''_j\}$ has infinitely many terms and the $\{x'_j\}$ has no terms. That completes the proof of part (iii).
(iv) Assume, for the sake of contradiction that there exists a nonoscillatory (about b) solution $\{x_n\}$ of equation (5)). We will consider the case when $x_n \in [h(b), b)$ for all $n \geq N_0$ and for some positive integer N_0. The case when $x_n \in [b, g(b)]$ for all $n \geq N_1$ for some positive integer N_1 is similar and it is omitted. Since $x_n \in [h(b), b)$ then

$$x_{n+1} = F(x_n) = g(x_n) > x_n;$$

so the sequence $\{x_n\}$ is eventually increasing and therefore convergent. Let $x = \lim\limits_{n\to\infty} x_n \in [h(b), b]$. Then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} g(x_n) = g(x),$$

which contradicts (H_3).
(v) Assume, for the sake of contradiction, that the solution of equation (5) $\{x_n\}$ converges to a limit x. Since all solutions of equation (5), other than those in the interval $[0, a)$, become trapped in an invariant interval $I = [h(b), g(b)]$, without loss of generality we may assume $x_n \in I$. Since $\{x_n\}$ is oscillatory, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ consisting of the last terms in every negative semicycle. Then $\{x_{n_k+1}\}$ is the subsequence of first terms in positive semicycles and $\lim\limits_{k\to\infty} x_{n_k} = x \leq b$.

On the other hand

$$x = \lim_{k\to\infty} x_{n_k+1} = \lim_{k\to\infty} F(x_{n_k}) = \lim_{k\to\infty} g(x_{n_k}) = g(x),$$

which is a contradiction because $g(x) > x$ for $x \in [a, b]$. The proof is complete. □

Next, we will define sequences $\{\alpha_n\}$ and $\{\beta_n\}$ which will be used in the study of oscillations and semicycles of solutions of equation (5).

Lemma 4. *Assume* $(H_1) - (H_4)$ *and* (7) *are satisfied. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be defined by*

$$\alpha_n = g^n(h(b)) \quad \textit{and} \quad \beta_n = h^n(g(b)), \quad n = 0, 1, \ldots \tag{8}$$

Then the following statements are true:

(i) *There exists a nonnegative integer* j *such that*

$$\alpha_0 < \alpha_1 < \ldots < \alpha_j < b \le \alpha_{j+1}. \tag{9}$$

(ii) *There exists a nonnegative integer* m *such that*

$$\beta_0 > \beta_1 > \ldots > \beta_m \ge b > \beta_{m+1}. \tag{10}$$

Proof. We only prove part (i). The proof of part (ii) is similar and will be omitted. Since $\alpha_0 = h(b) \in [a, b)$ we find $\alpha_1 = g(\alpha_0) > \alpha_0$. Furthermore, we have

$$\alpha_{i+1} = g(\alpha_i) > \alpha_i, \quad \text{provided } \alpha_i < b.$$

There are two possibilities. First, $\alpha_i < b$ for all $i = 0, 1, \ldots$. Then the sequence $\{\alpha_n\}$ is increasing and bounded from above by b, so it converges. Let $\alpha = \lim_{n\to\infty} \alpha_n$. Then $\alpha \in [h(b), b]$ and $\alpha = g(\alpha)$. This is a contradiction because the function g has no fixed points in $[h(b), b]$. So it remains that there exists a nonnegative integer j such that (9) holds. This completes the proof of the lemma.

Theorem 5. *Assume* $(H_1) - (H_4)$ *and* (7) *are satisfied. Let* $\{x_n\}$ *be a positive solution of equation* (5) *with initial conditions in* $[a, \infty)$. *Then the following statements are true*:

(i) *If* $g(h(b)) \ge b$, *then every negative semicycle relative to* b, *except perhaps the first one, has exactly one term.*

(ii) *If* $h(g(b)) \le b$, *then every positive semicycle relative to* b, *except perhaps the first one, has exactly one term.*

(iii) *If*

$$h(g(b)) \le b \le g(h(b)), \tag{11}$$

then every semicycle relative to b, *except perhaps the first one, has exactly one term.*

Proof. (i) Let x_j be the last term in a positive semicycle such that $x_j \in [b, h^{-1}(b))$. Then $x_{j+1} = F(x_j) = h(x_j) \geq h(b)$ and $x_{j+1} = h(x_j) < h(h^{-1}(b)) = b$, so $x_{j+1} \in [h(b), b) \subset [a, b)$ belongs to a negative semicycle. Furthermore, $x_{j+2} = F(x_{j+1}) = g(x_{j+1}) \geq g(h(b)) \geq b$ and $x_{j+2} = g(x_{j+1}) \leq g(b)$, so $x_{j+2} \in [b, g(b)]$ and it belongs to a positive semicycle. It remains to show that the last term of any positive semicycle always belongs to the interval $[b, h^{-1}(b))$. Assume for the sake of contradiction that $x_k \geq h^{-1}(b) > b$ is the last term of a positive semicycle. Then $x_{k+1} = F(x_k) = h(x_k) \geq h(h^{-1}(b)) = b$ also belongs to the positive semicycle. This is a contradiction because x_k is the last term of the positive semicycle.
(ii) The proof is similar and it is omitted.
(iii) It follows directly from (i) and (ii). □

The next two lemmas establish some properties and the character of semicycles of oscillatory solutions of equation (5).

Lemma 6. *Assume* $(H_1) - (H_4)$ *and* (7) *are satisfied and let*

$$g^j(h(b)) < b < g^{j+1}(h(b)), \quad \text{for some positive integer } j. \tag{12}$$

Furthermore, let $\{p_i\}_{i=0}^{j+1}$ *and* $\{q_i\}_{i=0}^{j+1}$ *be finite sequences defined by*

$$p_i = g^i(h(b)), \qquad q_i = g^{-j+i}(b), \qquad i = 0, 1, \ldots, j+1 \tag{13}$$

and let

$$P_i = [p_i, q_i), \quad i = 0, \ldots, j+1, \quad \text{and} \quad Q_i = [q_i, p_{i+1}), \quad i = 0, \ldots, j. \tag{14}$$

Then the following statements are true:
(i) $p_0 = h(b), q_j = b, q_{j+1} = g(b)$, *and the invariant interval for equation* (5) *is* $I = [p_0, q_{j+1}]$.
(ii) $p_{i+1} = g(p_i)$ *and* $q_{i+1} = g(q_i)$, *for* $i = 0, 1, \ldots, j$.
(iii) $p_i < q_i < p_{i+1} < q_{i+1}$, *for* $i = 0, 1, \ldots, j$.
(iv) $P_i \cap Q_j = \emptyset$, $i \neq j$, *and* $\left(\bigcup_{i=0}^{j+1} P_i\right) \cup \left(\bigcup_{i=0}^{j} Q_i\right) \cup \{q_{j+1}\} = I$.

Proof. Parts (i) and (ii) follow directly from (13).
(iii) Since the function g is increasing, g^{j-i} is also increasing. The statement

$$g^i(h(b)) = p_i < q_i = g^{-j+i}(b)$$

is equivalent to

$$g^j(h(b)) = g^{j-i}(g^i(h(b)) = g^{j-i}(p_i) < g^{j-i}(q_i) = g^{j-i}(g^{-j+i}(b)) = b$$

which is true because of (12). Similarly, the inequality

$$g^{-j+i}(b) = q_i < p_{i+1} = g^{i+1}(h(b))$$

is equivalent to

$$b = g^{j-i}(g^{-j+i}(b)) = g^{j-i}(q_i) < g^{j-i}(p_{i+1}) = g^{j-i}(g^{i+1}(h(b))) = g^{j+1}(h(b))$$

which also follows from (12). Finally, (iv) follows from (i)-(iii) and (14).

Lemma 7. *Assume* $(H_1) - (H_4)$, (7), *and* (12) *are satisfied and let the sequences* $\{p_i\}_{i=0}^{j+1}$ *and* $\{q_i\}_{i=0}^{j+1}$ *and intervals* P_i, Q_i *be defined by* (13) *and* (14), *respectively. Let* $\{x_n\}$ *be a solution of equation* (5) *with initial condition in* $[a, \infty)$. *Then the following statements are true*:

(i) *If* $x_n \in P_i$, *then* $x_{n+1} \in P_{i+1}, \quad i = 0, 1, \ldots, j.$

(ii) *If* $x_n \in Q_i$, *then* $x_{n+1} \in Q_{i+1}, \quad i = 0, 1, \ldots, j-1.$

(iii) *If* $x_n \in P_{j+1}$, *then* $x_{n+1} \in [p_0, h(g(b)))$.

(iv) *If* $x_n \in Q_j$, *then* $x_{n+1} \in [p_0, h(g(b)))$.

(v) *If* $x_n \in P_0$, *then*

$$\begin{aligned} x_{n+i} &\in P_i \subset [h(b), b), \quad i = 0, 1, \ldots, j, \\ x_{n+j+1} &\in P_{j+1} \subset [b, g(b)), \\ x_{n+j+2} &\in [p_0, h(g(b))). \end{aligned}$$

(vi) *If* $x_n \in Q_0$, *then*

$$\begin{aligned} x_{n+i} &\in Q_i \subset [h(b), b), \quad i = 0, 1, \ldots, j-1, \\ x_{n+j} &\in Q_j \subset [b, g(b)), \\ x_{n+j+1} &\in [p_0, h(g(b))). \end{aligned}$$

(vii) *If* $x_n = q_{j+1}$, *then* $x_{n+1} = h(g(b))$.

Proof. (i) Since $a < p_i \le x_n < q_i \le b$, where, from Lemma 6,

$$a < h(b) = p_0 \le p_i = g^i(h(b)) < b \quad \text{for} \quad i = 0, 1, \ldots, j,$$

and where

$$a < p_i < q_i = g^{-j+i}(b) < g^{-j+i+1}(b) \le b \quad \text{for} \quad i = 0, 1, \ldots, j-1,$$

then

$$p_{i+1} = g(p_i) \le x_{n+1} = g(x_n) < g(q_i) = q_{i+1}.$$

(ii) The proof is similar to (i) and will be omitted.

(iii) Since $x_n \ge p_{j+1} = g^{j+1}(h(b)) > b$, then

$$x_{n+1} = F(x_n) = h(x_n) > h(b) = p_0.$$

Furthermore $x_n < q_{j+1} = g(b)$ implies

$$x_{n+1} = F(x_n) = h(x_n) < h(g(b)).$$

(iv) Clearly, $x_n \ge q_j = b$ implies

$$x_{n+1} = F(x_n) = h(x_n) > h(b) = p_0.$$

Next from $g^j(h(b)) < b$ it follows that

$$x_n < p_{j+1} = g^{j+1}(h(b)) = g(g^j(h(b))) \leq g(b),$$

and similarly as in (iii) we obtain $x_{n+1} < h(g(b))$.
(v) From (i) it follows that $x_n \in P_0$ implies $x_{n+1} \in P_1$, $x_{n+2} \in P_2$, $\ldots$, $x_{n+j} \in P_j$, and $x_{n+j+1} \in P_{j+1}$. Furthermore, from (iii) we get $x_{n+j+2} \in [p_0, h(g(b)))$. The proof of (vi) is similar to (v) and is omitted. The proof of (vii) is trivial. □

The following two lemmas are analogs to lemmas 6 and 7 so we formulate them without proof.

Lemma 8. *Assume* $(H_1) - (H_4)$ *and* (7) *are satisfied and let*

(15) $$h^{m+1}(g(b)) < b < h^m(g(b)), \quad \text{for some positive integer } m.$$

Furthermore, let $\{r_i\}_{i=0}^{m+1}$ *and* $\{s_i\}_{i=0}^{m+1}$ *be finite sequences defined by*

(16) $$r_i = h^{-m+i}(b), \qquad s_i = h^i(g(b)), \qquad i = 0, 1, \ldots, m+1,$$

and let $R_i = [r_i, s_i)$, $i = 0, \ldots, m+1$, *and* $S_j = [s_{j+1}, r_j)$, $j = 0, \ldots, m$.
Then the following statements are true:

(i) $r_{m+1} = h(b)$, $\quad r_m = b$, $\quad s_0 = g(b)$, *so the invariant interval for equation* (5) *is* $I = [r_{m+1}, s_0]$.

(ii) $r_{i+1} = h(r_i)$ *and* $s_{i+1} = h(s_i)$, *for* $i = 0, 1, \ldots, m$.

(iii) $r_{i+1} < s_{i+1} < r_i < s_i$, *for* $i = 0, 1, \ldots, m$.

(iv) $R_i \cap S_j = \emptyset$, $i \neq j$ *and* $\left(\bigcup_{i=0}^{m+1} R_i\right) \cup \left(\bigcup_{i=0}^{m} S_i\right) \cup \{s_0\} = I$.

Lemma 9. *Assume* $(H_1)-(H_4)$ *and* (7) *are satisfied and let the sequences* $\{r_i\}_{i=0}^{m+1}$ *and* $\{s_i\}_{i=0}^{m+1}$ *be defined by* (16). *Let* $\{x_n\}$ *be a solution of equation* (5) *with initial condition in* $[a, \infty)$. *Then the following statements are true*:

(i) *If* $x_n \in R_i$, *then* $x_{n+1} \in R_{i+1}$, $\quad i = 0, 1, \ldots, m$.

(ii) *If* $x_n \in S_i$, *then* $x_{n+1} \in S_{i+1}$, $\quad i = 0, 1, \ldots, m-1$.

(iii) *If* $x_n \in R_{m+1}$, *then* $x_{n+1} \in [g(h(b)), s_0)$.

(iv) *If* $x_n \in S_m$, *then* $x_{n+1} \in [g(h(b)), s_0)$.

(v) *If* $x_n \in R_0$, *then*

$$x_{n+i} \in R_i \subset [b, g(b)), \quad i = 0, 1, \ldots, m,$$
$$x_{n+m+1} \in R_{m+1} \subset [h(b), b),$$
$$x_{n+m+2} \in [g(h(b)), s_0).$$

(vi) *If* $x_n \in S_0$, *then*

$$x_{n+i} \in S_i \subset [b, g(b)), \quad i = 0, 1, \ldots, m-1,$$
$$x_{n+m} \in S_m \subset [h(b), b),$$
$$x_{n+m+1} \in [g(h(b)), s_0).$$

(vii) *If* $x_n = s_0$, *then* $x_{n+1} = h(g(b))$.

The following theorem follows directly from lemmas 7 and 9. It provides the detailed description of the structure of semicycles of equation (5).

Theorem 10. *Assume* $(H_1)-(H_4)$ *and* (7) *are satisfied and let* $\{x_n\}$ *be a solution of equation* (5) *with initial condition in* $[a,\infty)$. *Then the following statements are true*:

(i) *If* (12) *holds, then every negative semicycle, except perhaps the first one, has length at most* $j+1$.

(ii) *If* (15) *holds, then every positive semicycle, except perhaps the first one, has length at most* $m+1$.

The following theorem establishes some sufficient conditions for the existence of periodic orbits of equation (5).

Theorem 11. *Assume* $(H_1)-(H_4)$ *and* (7) *are satisfied. Then the following statements are true:*

(i) *If* $g^j(h(b))=b$, *then the solution* $\{x_n\}$ *of equation* (5) *with initial condition* $x_0=b$ *is periodic with period* $j+1$.

(ii) *If* $h^m(g(b))=b$, *then the solution* $\{x_n\}$ *of equation* (5) *with initial condition* $x_0=b$ *is periodic with period* $m+1$.

Proof. (i) Clearly, if $x_0=b$, then

$$\begin{aligned} x_1 &= F(x_0)=F(b)=h(b)<b,\\ x_2 &= F(x_1)=g(h(b))<b,\ldots,\\ x_{j+1} &= F(x_j)=g^j(h(b))=b=x_0 \end{aligned}$$

and the proof is complete. The proof of (ii) is similar and it is omitted. □

The next theorem gives sufficient conditions for the existence and attractivity of period-2 solutions of equation (5).

Theorem 12. *Assume* $(H_1)-(H_4)$ *and* (7) *are satisfied. Let* $\{x_n\}$ *be a positive solution of equation* (5) *with initial conditions in* $[a,\infty)$. *Then the following statements are true*:

(i) *If* (11) *is satisfied, then the equation* (5) *has at least two period-*2 *solutions* $\{\bar{x},h(\bar{x}),\bar{x},h(\bar{x}),\ldots\}$ *and* $\{h(\bar{x}),\bar{x},h(\bar{x}),\bar{x},\ldots\}$, *where* $\bar{x}$ *is a fixed point of the function* $k(x)=g(h(x))$ *in the interval* $[b,g(b)]$.

(ii) *If* $\bar{x}$ *is the unique fixed point of the function* k *in the interval* $[b,g(b)]$, *then the following statements are true*:

(a) *the period-*2 *solution* $\{\bar{x},h(\bar{x}),\bar{x},h(\bar{x}),\ldots\}$ *attracts all solutions with initial conditions in* $[b,g(b)]$;

(b) *the period-*2 *solution* $\{h(\bar{x}),\bar{x},h(\bar{x}),\bar{x},\ldots\}$ *attracts all solutions with initial conditions in* $[h(b),b)$;

(c) *if* $h(b)>a$ *then all solutions with initial conditions in* $[a,h(b))\cup(g(b),\infty)$ *are* *attracted to either* $\{\bar{x}, h(\bar{x}), \bar{x}, h(\bar{x}), \ldots\}$ *or* $\{h(\bar{x}), \bar{x}, h(\bar{x}), \bar{x}, \ldots\}$ *period-2 solutions;*

(d) *if* $h(b) = a$ *then all solutions with initial conditions in* $(g(b), \infty)$ *are attracted to either* $\{\bar{x}, h(\bar{x}), \bar{x}, h(\bar{x}), \ldots\}$ *or* $\{h(\bar{x}), \bar{x}, h(\bar{x}), \bar{x}, \ldots\}$ *period-2 solutions;*

(e) *all solutions of equation* (5) *with initial conditions in* $[a, \infty)$ *are attracted to either* $\{\bar{x}, h(\bar{x}), \bar{x}, h(\bar{x}), \ldots\}$ *or* $\{h(\bar{x}), \bar{x}, h(\bar{x}), \bar{x}, \ldots\}$ *period-2 solutions.*

Proof. (i) First, we will show that the function k, defined by $k(x) = g(h(x))$ maps the interval $[b, g(b)]$ into itself. Since both g and h are increasing, by using (11), we find

$$h([b, g(b)]) = [h(b), h(g(b))] \subset [h(b), b]$$

and

$$k([b, g(b)]) = g(h([b, g(b)])) \subset g([h(b), b]) = [g(h(b)), g(b)] \subset [b, g(b)].$$

Since g and h are continuous on intervals $[h(b), b] \subset [a, b]$ and $[b, g(b)] \subset [b, \infty)$, respectively, the composite function k is also continuous on $[b, g(b)]$. Then, according to Brouwer's Fixed Point Theorem, the function k has at least one fixed point $\bar{x}$ in $[b, g(b)]$.

(ii) (a) Let $x_0 \in [b, g(b)]$. Consider the subsequence $\{x_{2i}\}$ of even numbered terms of the corresponding solution $\{x_n\}$. Clearly

$$x_{2i+2} = F(x_{2i+1}) = F(F(x_{2i})) = g(h(x_{2i})) = k(x_{2i}), \quad i = 0, 1, \ldots.$$

Since the function k maps $[b, g(b)]$ into itself, it is increasing, and it is assumed that k has the unique fixed point $\bar{x}$ in $[b, g(b)]$, by applying Theorem B, we obtain that $\bar{x}$ is the attractor of all subsequences $\{x_{2i}\}$ of solutions $\{x_n\}$ of equation (5) with initial conditions in $[b, g(b)]$. To complete the proof we have to show that $h(\bar{x})$ attracts all subsequences $\{x_{2i+1}\}$ of odd-numbered terms of the solution $\{x_n\}$ of equation (5). Since h is a continuous function on $[b, g(b)]$, and $x_{2i+1} = h(x_{2i})$, $i = 0, 1, \ldots,$ we obtain

$$\lim_{i\to\infty} x_{2i+1} = \lim_{i\to\infty} h(x_{2i}) = h(\bar{x}),$$

which completes the proof of part (ii)(a).

(ii)(b) The proof is similar to the proof of part (ii)(a) and it is omitted.

(ii)(c) Let $x_0 \in [a, h(b)) \cup (g(b), \infty) \subset [a, \infty)$. From Theorem 3(iv), It follows that the corresponding solution $\{x_n\}$ will be eventually trapped in an invariant interval $I = [h(b), g(b)]$. Let i be the smallest positive integer such that $x_{i-1} \in [a, h(b)) \cup (g(b), \infty)$, and $x_i \in I = [h(b), g(b)]$. Then $x_n \in I = [h(b), g(b)]$, for $n = i, i+1, \ldots$. Also, x_i belongs to either $[h(b), b)$ or $[b, g(b)]$. Therefore, if $x_i \in [b, g(b)]$ the corresponding solution $\{x_n\}$ will be attracted to $\{\bar{x}, h(\bar{x}), \bar{x}, h(\bar{x}), \ldots\}$, or if $x_i \in [h(b), b)$, the solution is attracted to $\{h(\bar{x}), \bar{x}, h(\bar{x}), \bar{x}, \ldots\}$.

(ii)(d) The proof is similar to the case (ii)(c) and it is omitted.

(ii)(e) It follows directly from parts (ii)(a)-(d) and the fact that

$$[a,\infty) = [a,h(b)) \cup [h(b),b) \cup [b,g(b)) \cup [g(b),\infty), \qquad \text{when } h(b) > a$$

or

$$[a,\infty) = [a,b) \cup [b,g(b)) \cup [g(b),\infty), \qquad \text{when } h(b) = a$$

which completes the proof of the theorem.

3. THE CASE $h(b) < a$

In this section we will examine the behavior of solutions of equation (5) in the case when

$$h(b) < a. \tag{17}$$

Lemma 13. *Assume* $(H_1) - (H_4)$ *and* (17) *are satisfied. Then the following statements are true*:

(i) *If* $x_0 \in [b,\infty)$, *then there exists a positive integer* k *such that*

$$x_0 > x_1 > \ldots > x_{k-1} \geq b > x_k.$$

(ii) *If* $x_0 \in [a,b)$, *then there exists a positive integer* m *such that*

$$x_0 < x_1 < \ldots < x_{m-1} < b < x_m.$$

Proof. We only prove part (i). The proof of (ii) is similar and is omitted. First of all, if $x_0 = b$, then by (17) $x_1 = h(x_0) = h(b) < a < b$, and we are done. So, suppose that $x_0 \in (b,\infty)$. Assume for the sake of contradiction $x_k \in (b,\infty)$, for every nonnegative integer k. Then $x_k \in (b,\infty)$, implies which $x_{k+1} = F(x_k) = h(x_k) < x_k$, so the sequence $\{x_n\}$ is decreasing and bounded from below by b. Thus $\{x_n\}$ converges to a limit $x \in [b,\infty)$. Since

$$x_{n+1} = F(x_n) = h(x_n)$$

by letting $n \to \infty$ we obtain $x = h(x) < x$, which contradicts (H_3), and the proof is complete. □

The following lemma provides sufficient conditions when 0 is the global attractor of all solutions.

Lemma 14. *Assume* $(H_1) - (H_4)$ *and* (17) *are satisfied and let* $H \leq a$. *Then all solutions of equation* (5) *converge to* 0.

Proof. If $x_0 \in [0,a)$, then from Lemma 1 it follows that the corresponding solution $\{x_n\}$ converges to 0. If $x_0 \in [b,\infty)$, then $x_1 = F(x_0) = h(x_0) < H \leq a$ and so $x_1 \in (0,a)$ and the corresponding solution converges to 0. Finally, if $x_0 \in [a,b)$, from Lemma 13(ii), it follows that there exists a positive integer k such that $x_k \in [b,\infty)$; so $x_{k+1} \in (0,a)$, and $\{x_n\}$ converges to 0. □

The following technical lemma follows directly from the fact that the function h is increasing and has an inverse h^{-1} and Lemma 1. The proof is trivial and also is omitted.

Lemma 15. *Assume* $(H_1)-(H_4)$ *and* (17) *are satisfied and let* $a < H \leq b$. *Then the following statements are true*:

(i) *If* $x_0 \in [b, h^{-1}(a))$, *then* $x_1 \in [h(b), a)$ *and the corresponding solution* $\{x_n\}$ *converges to* 0.

(ii) *If* $x_0 \in [h^{-1}(a), \infty)$, *then* $x_1 \in [a, H) \subset [a, b)$.

The next technical lemma will be useful in the sequel.

Lemma 16. *Assume* $(H_1)-(H_4)$, (17), *and*

$$H > b \tag{18}$$

are satisfied. Then the following statements are true:

(i) *The function* h^{-1}, *an inverse to* h, *is defined on the interval* $[h(b), H)$, *and it is increasing and satisfies* $h^{-1}(x) > x$, *for* $x \in [h(b), H)$.

(ii) *There exists a positive integer* k *such that*

$$a < h^{-1}(a) < h^{-2}(a) < \ldots < h^{-k+1}(a) \leq H < h^{-k}(a)$$

and

$$h^{-i+1}(b) < h^{-i}(a) < h^{-i}(b), \quad i = 1, \ldots, k.$$

Proof. The proof of part (i) is trivial and it is omitted. To prove part (ii) assume for the sake of contradiction $h^{-k}(a) \leq H$, for every positive integer k. Since $a < h^{-1}(a)$, it follows that

$$h^{-k+1}(a) < h^{-k+1}(h^{-1}(a)) = h^{-k}(a)$$

and the sequence $\{h^{-n}(a)\}$ is increasing and bounded from above by H. Therefore it converges to a limit $\bar{a} \leq H$. Since $h^{-n+1}(a) = h(h^{-n}(a))$ by letting $n \to \infty$, we obtain $h(\bar{a}) = \bar{a} < \bar{a}$, which is contradiction. The remaining part follows directly from $h(b) < a < b$. □

Next, to understand the behavior of solutions of equation (5) with initial conditions in $[b, \infty)$, we will introduce the following intervals, provided that their endpoints are defined. Let

$$I_i^h = [h^{-i}(b), h^{-i-1}(a)), \quad \text{and} \quad J_i^h = [h^{-i-1}(a), h^{-i-1}(b)), \quad i = 0, 1, \ldots. \tag{19}$$

The following lemma summarizes some properties of intervals I_i^h and J_i^h.

Lemma 17. *Assume* $(H_1)-(H_4)$, *(17), and (18) are satisfied and let intervals* I_i^h *and* J_i^h *be given by (19) provided they are well defined. Then the following statements are true*:

(i) $I_i^h, J_i^h \subset [b, \infty), I_i^h \cap I_j^h = \emptyset$ *and* $J_i^h \cap J_j^h = \emptyset$, *for* $i, j = 0, 1, \ldots$ *and* $i \neq j$.

(ii) $I_i^h \cap J_j^h = \emptyset$ *for* $i, j = 0, 1, \ldots$.

(iii) *There exists a positive integer* k *such that either* $H \in I_{k-1}^h$ *or* $H \in J_{k-2}^h$.

(iv) *For* k *from part* (iii)

$$F(I_i^h) = I_{i-1}^h, \quad F(J_i^h) = J_{i-1}^h, \quad i = 1, \ldots, k-1,$$
$$F(I_0^h) = F([b, h^{-1}(a)) = [h(b), a) \subset [0, a),$$
$$F(J_0^h) = F([h^{-1}(a), h^{-1}(b)) = [a, b).$$

(v) *If* $H \in I_{k-1}^h$, *for some positive integer* k, *then* $F([h^{-k}(b), \infty)) = [h^{-k+1}(b), H) \subset I_{k-1}^h$.

(vi) *If* $H \in J_{k-2}^h$, *for some positive integer* k, *then* $F([h^{-k}(a), \infty)) = [h^{-k+1}(a), H) \subset J_{k-2}^h$.

(vii) *For* k *from part* (iii)

$$\bigcup_{i=0}^{k-1} (I_i^h \cup J_i^h) \subset [b, \infty) \quad \textit{and} \quad F([h^{-1}(b), \infty)) \subset \bigcup_{i=0}^{k-1} (I_i^h \cup J_i^h).$$

Proof. The proof of parts (i) and (ii) follows directly from the definition of intervals I_i^h and J_i^h and the fact that h^{-1} is an increasing function. Part (iii) follows directly from Lemma 16 (ii). The proof of part (iv) follows from the definition of intervals I_i^h and J_i^h and the facts that $I_i^h, J_i^h \subset [b, \infty)$ and $F(x) = h(x)$, for $x \in [b, \infty)$. Next, for part (v), since $H \in I_{k-1}^h$, then $h^{-k+1}(b) \leq H < h^{-k}(a)$, and so

$$F([h^{-k}(b), \infty)) = [h^{-k+1}(b), H) \subset [h^{-k+1}(b), h^{-k}(a)) \subset I_{k-1}^h.$$

The proof of part (vi) is similar to the proof of part (v) and it is omitted. Finally, part (vii) follows from the definition of intervals I_i^h and J_i^h and parts (i) and (iv) - (vi). This completes the proof of the lemma. □

The next theorem describes the behavior of solutions with their initial condition in $[b, \infty)$. The proof follows directly from the previous lemma and it is omitted.

Theorem 18. *Assume* $(H_1) - (H_4)$, (17), *and* (18) *are satisfied and let intervals* I_i^h *and* J_i^h *be given by* (19). *For the initial condition* $x_0 \in [b, \infty)$ *the following statements are true*:

(i) *Let* $H \in I_{k-1}^h$ *for some positive integer* k. *Then*

(a) *if* $x_0 \in [h^{-k}(b), \infty)$, *then*

$$x_1 \in I_{k-1}^h, x_2 \in I_{k-2}^h, \ldots, x_k \in I_0^h \quad \textit{and} \quad x_{k+1} \in [h(b), a),$$

and the corresponding solution $\{x_n\}$ *converges to* 0;

(b) *if* $x_0 \in I_{m-1}^h$, *where* $m \leq k$, *then*

$$x_1 \in I_{m-2}^h, x_2 \in I_{m-3}^h, \ldots, x_{m-1} \in I_0^h \quad \textit{and} \quad x_m \in [h(b), a),$$

and the corresponding solution $\{x_n\}$ *converges to* 0;

(c) *if* $x_0 \in J^h_{m-1}$, *where* $m \le k$, *then*

$$x_1 \in J^h_{m-2}, x_2 \in J^h_{m-3}, \ldots, x_{m-1} \in J^h_0 \quad \textit{and} \quad x_m \in [a,b).$$

(ii) *Let* $H \in J^h_{k-2}$ *for some positive integer* k. *Then*

(a) *if* $x_0 \in [h^{-k-1}(a), \infty)$, *then*

$$x_1 \in J^h_{k-1}, x_2 \in J^h_{k-1}, \ldots, x_k \in J^h_0 \quad \textit{and} \quad x_{k+1} \in [a,b);$$

(b) *if* $x_0 \in J^h_{m-1}$, *where* $m \le k$, *then*

$$x_1 \in J^h_{m-2}, x_2 \in J^h_{m-3}, \ldots, x_{m-1} \in J^h_0 \quad \textit{and} \quad x_m \in [a,b);$$

(c) *if* $x_0 \in I^h_{m-1}$, *where* $m \le k$, *then*

$$x_1 \in I^h_{m-2}, x_2 \in I^h_{m-3}, \ldots, x_{m-1} \in I^h_0 \quad \textit{and} \quad x_m \in [h(b), a),$$

and the corresponding solution $\{x_n\}$ *converges to* 0.

Next we will focus on the case when initial conditions are in the interval $[a,b)$.

Theorem 19. *Assume* $(H_1)-(H_4)$ *and* (17) *are satisfied and let* $x_0 \in [a,b)$. *Then there exists a positive integer* k *such that*

$$a \le x_0 < x_1 < \ldots < x_{k-1} < b \le x_k.$$

Furthermore, either there exists a positive integer ℓ *such that* $x_{k+\ell} \in [h(b), a)$ *or there exists a positive integer* m *such that* $x_{k+m} \in [a,b)$.

Proof. Assume for the sake of contradiction $x_k \in [a,b)$ for every nonnegative integer k. Then $x_k \in [a,b)$ implies $x_{k+1} = F(x_k) = g(x_k) > x_k$, so the sequence $\{x_n\}$ is increasing and bounded from above by b, and therefore converges to a limit $x \in [a,b]$. Since $x_{n+1} = F(x_n) = g(x_n)$, by letting $n \to \infty$, we obtain $x = g(x) > x$, which is a contradiction. The remaining part follows directly from Theorem 18, which completes the proof.

4. EXAMPLE

In this section we present an example which illustrates the previous results.

Consider the difference equation

$$x_{n+1} = F(x_n), \quad n = 0, 1, \ldots, \tag{20}$$

where $x_0 > 0$ and the function F is defined by

$$F(x) = \begin{cases} x^2/2, & \text{if } x \in [0,1) \\ 2\sqrt{2x}, & \text{if } x \in [1,2) \\ \alpha x/(1+x), & \text{if } x \in [2,\infty) \end{cases} \tag{21}$$

with $\alpha > 0$. So in our case we have

$$f(x) = \frac{1}{2}x^2,\ g(x) = 2\sqrt{2x},\ h(x) = \frac{\alpha x}{1+x},\ a = 1,\ b = 2,\ H = \lim_{x\to\infty} h(x) = \alpha, \tag{22}$$

and hypotheses (H_1) - (H_4) are satisfied provided $\alpha < 3$. Condition (7) becomes

$$\alpha > 3/2. \tag{23}$$

In this example we have

$$h(g(b)) = \frac{\alpha g(b)}{1+g(b)} = \frac{4\alpha}{5} \quad \text{and} \quad g(h(b)) = 2\sqrt{2h(b)} = \frac{4\sqrt{3\alpha}}{3}.$$

Therefore condition (11) becomes $3/4 \le \alpha \le 5/2$. The following corollaries summarize the properties of solutions of equation (20)-(21). They follow from Lemmas 1, 14, and 15 and Theorems 3, 5, 10, 11, 12, and 19.

Corollary 20. *Consider equation* (20), *with F is defined by* (21), *and assume*

$$3/2 < \alpha < 3. \tag{24}$$

Then the following statements are true:

(i) *If $x_0 \in (0,1)$, then the corresponding solution decreases and converges to* 0.

(ii) *Equation* (20) *with* (21) *has the invariant interval $I = [2\alpha/3, 4]$ and all solutions with initial conditions in $[1,\infty)$ become trapped in the invariant interval I.*

(iii) *All solutions with initial conditions in $[1,\infty)$ strictly oscillate about* 2 *and no solution converges.*

(iv) *If*

$$3/2 < \alpha < 5/2, \tag{25}$$

then every semicycle, except perhaps the first one, of the solution with initial condition in $[1,\infty)$, has length 1. *Furthermore, there exist two period-2 cycles,*

$$C_1' = \{x', x'', x', x'', \ldots\} \text{ and } C_2'' = \{x'', x', x'', x', \ldots\}$$

where $x' = \dfrac{-1+\sqrt{1+32\alpha}}{2} \in [2,\infty)$ and $x'' = \alpha - \dfrac{x'}{8} \in [1,2)$.

(v) *Every solution of the given equation with initial conditions in $[1,\infty)$ is attracted to either C_2' or C_2''.*

(vi) *If $\alpha = 5/2$, then there are two attractive period-*2 *cycles*

$$C_1' = \{4,2,4,2,\ldots\} \text{ and } C_2'' = \{2,4,2,4,\ldots\}.$$

The following table provides the intervals for parameter α and the corresponding maximum length of semicycles. These numerical values of endpoints of all intervals, except the first one, are obtained by numerically solving the equations

$$h^m(g(2)) = 2, \quad \text{where} \quad m = 1, 2, \ldots,$$

for α. The first interval follows from condition (25).

Interval for α; [equivalent to condition (15)]	Max length of semicycles	
	Negat.	Posit.
$(1.5, 2.5)$	1	1
$(2.5, 2.87083)$	1	2
$(2.87083, 2.96071)$	1	3
$(2.96071, 2.98736)$	1	4
$(2.98736, 2.99585)$	1	5
$(2.99585, 2.99862)$	1	6
$(2.99862, 2.99954)$	1	7
$(2.99954, 2.99985)$	1	8
$(2.99985, 2.99995)$	1	9
$(2.99995, 2.99998)$	1	10
$(2.99998, 2.99999)$	1	11

Table 1. Lengths of semicycles in terms of α

n	x_n	Semicycle
10	1.6838956235	−
11	3.6703085685	+
12	1.9804210915	−
13	3.9803729388	+
14	2.0140137956	+
15	1.6839056186	−
16	3.6703194614	+
17	1.9804223499	−
18	3.9803742035	+
19	2.0140139240	+
20	1.6839056542	−
21	3.6703195002	+
22	1.9804223544	−
23	3.9803742080	+
24	2.0140139245	+

Table 2. Time series for $\alpha = 2.52$

The next result describes the behavior of solutions when $\alpha < 3/2$.

Corollary 21. *Consider equation* (20), *with* F *defined by* (21), *and assume*

$$\alpha < 3/2. \tag{26}$$

Then the following statements are true:

(i) *If* $\alpha \leq 1$, *then all solutions of the given equation converge to* 0.

(ii) *Let* $1 < \alpha < 3/2$. *If* $x_0 \in [2, 1/(\alpha-1))$, *then* $x_1 \in [2\alpha/3, 1)$ *and the corresponding solution converges to* 0. *If* $x_0 \in [1/(\alpha-1), \infty)$, *then* $x_1 \in [1, \alpha) \subset [1, 2)$.

(iii) *If* $x_0 \in [1, 2)$, *then there exists a positive integer* k *such that*

$$1 \leq x_0 < x_1 < \ldots < x_{k-1} < 2 \leq x_k.$$

Furthermore, either there exists a positive integer ℓ *such that* $x_{k+\ell} \in [2\alpha/3, 1)$ *or there exists a positive integer* m *such that* $x_{k+m} \in [1, 2)$.

Time series of the solution $\{x_n\}$, for different values of α are shown on Figure 1. Note that for $\alpha = 1.25$ the corresponding solution converges to 0. Also, for $\alpha = 2.52$ positive semicycles have length either 1 or 2 while for $\alpha = 2.75$, positive semicycles have length 2. Both values of α belong to the interval where the maximum length of positive semicycles is 2.

In all cases, except for $\alpha = 1.25$, the terms where we have "peaks" are the greatest terms in positive semicycles. All terms between two "peaks" belong to the corresponding positive semicycle, except the last one (the one which precedes the next "peak") which belongs to the negative semicycle. In some cases, as,

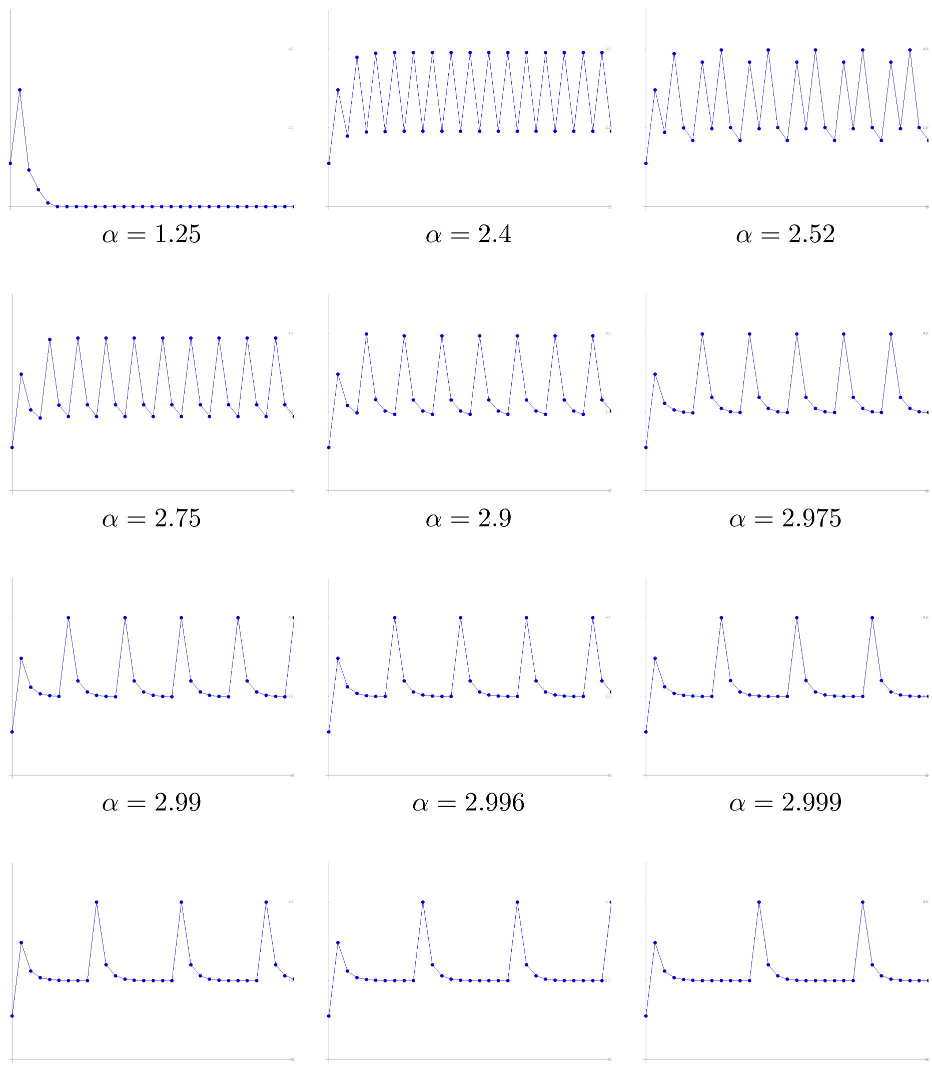

Figure 1. Time series of $\{x_n\}$ for $n = 0, 1, \ldots, 30$, and different values of parameter α; initial condition $x_0 = 1.1$.

for example, when $\alpha = 2.52$, the graph could be slightly misleading due to the fact that some terms in positive and negative semicycles are very close to 2 in value.

Table 2 contains values of terms from consecutive few semicycles which additionally clarifies the observations and is in accordance to the theoretical result that positive semicycles have length 1 or 2, while negative semicycles have length 1.

The bifurcation diagram for $\alpha \in (0,3)$ is given in Figure 2. According to Corollaries 20 and 21 the attractive period-2 cycle exists for $\alpha \in (1.5, 2.5]$, while for $\alpha \in (0,1]$, all positive solutions converge to 0. However, for $\alpha \in (1, 1.5]$, there is no conclusion about the asymptotic behavior of solutions. From the bifurcation diagram one can observe that there exists a value $\alpha_{cr} \in (1, 1.5]$ where the behavior of the solution changes from convergence to 0 to convergence to period-2 cycle.

Experimentally we found a critical value for α to be $\alpha_{cr} \approx 1.3535533906$.

Also, from the bifurcation diagram we observe that for $\alpha > 2.5$ and close to 2.5 there exists an attractive period-5 cycle and when α increases, an attractive period-3 cycle appears.

Actually, for $\alpha > 2.5$ the situation is more complex. Namely, for $\alpha = 2.544$, an attractive period-3 exists and when α decreases to 2.5, the equation undergoes series of bifurcations where attractive periodic cycles of the periods 5, 7, 9, 11, and so on appear. As α approaches the value 2.5 from the right, periods of the attractive cycles increases. Finally, when α crosses the value 2.5, the attractive period-2 cycle appears. Figure 3 illustrates the sequence of bifurcations showing the time series for attractive periodic cycles.

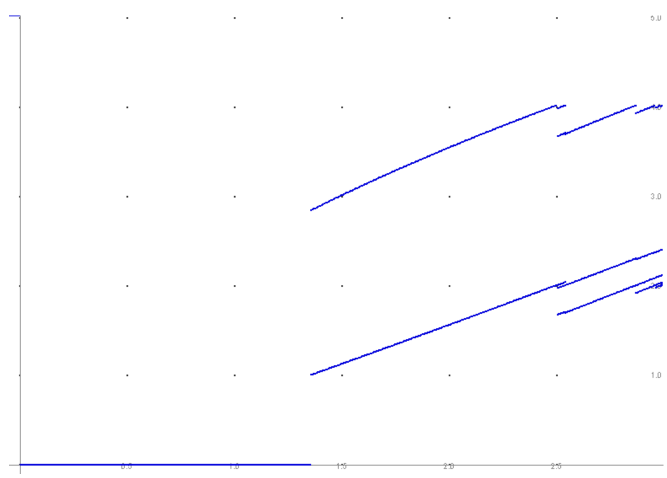

Figure 2. Bifurcation diagram of equation (20) with (21) for $\alpha \in (0,3)$. Values x_n, for $500 \leq n \leq 1000$, are ploted for 500 values of α.

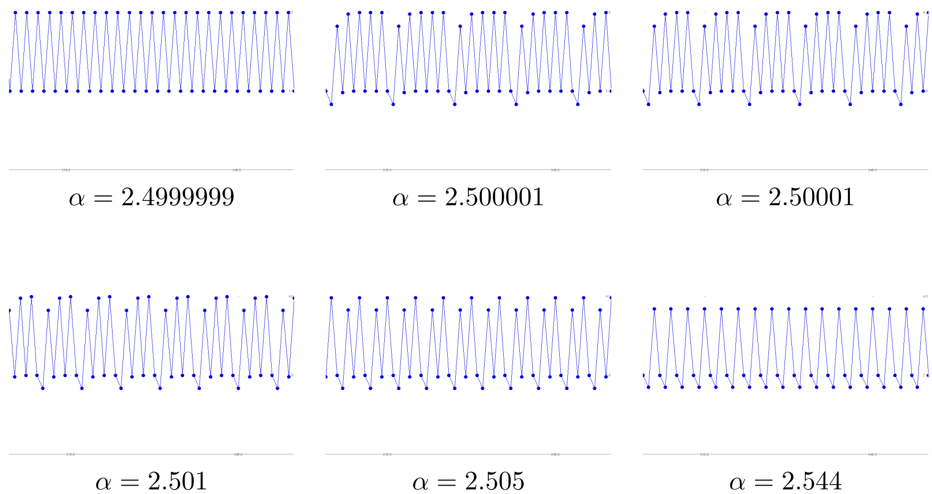

Figure 3. Time series for $\{x_n\}$, $200 \leq n \leq 250$ of $\{x_n\}$ indicate the presence of attractive periodic cycles for different values of α

Attractive period-3 cycles exist for $\alpha \in (2.544, 2.878286)$ where the interval is experimentally determined. When α decreases to 2.878286 again a series of bifurcations takes place where attractive periodic cycles of periods 4, 7, 10, 13, 16 and so on appear. Figure 4 illustrates the described behavior.

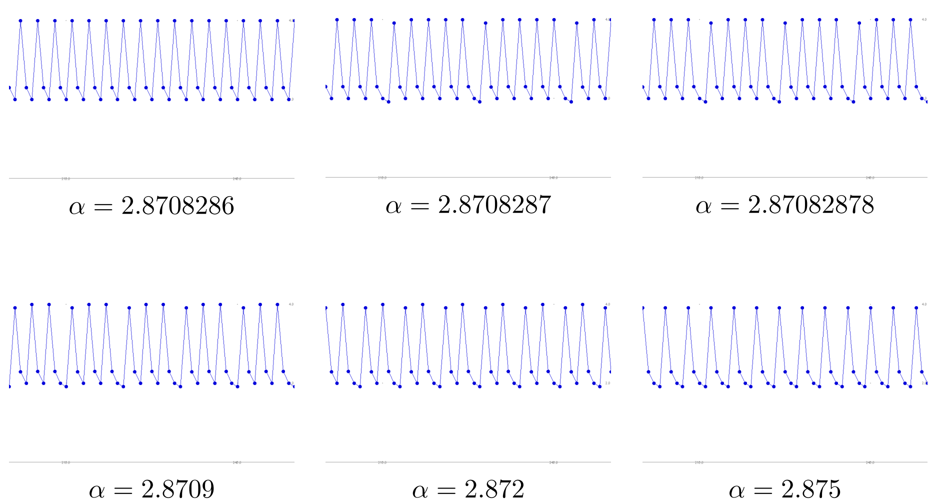

Figure 4. Time series for $\{x_n\}$, $200 \leq n \leq 250$ of $\{x_n\}$ indicate the presence of attractive periodic cycles for different values of α

5. CONCLUDING REMARKS

In the case when condition (7) holds equation (5) with (6) exhibits the characteristics of the Modified Allee Effect. Namely, if the initial conditions are from $(0, a)$, according to Lemma 1, all solutions decrease to 0. Also, from Theorem 3, it follows that all solutions with initial conditions from $[a, \infty)$, become eventually trapped in the invariant interval I. However, in the case when condition (17) holds, the situation is more complex. Again, when initial conditions are in $(0, a)$, the equilibrium 0 attracts corresponding solutions. But with initial conditions in $[a, \infty)$, the situation is not clear. It is possible that some solutions are attracted to 0, while others remain in the interval $[a, \infty)$. Lemmas 14, 15, and 16 and Theorems 18 and 19 are good illustrations of the complexity of the dynamics in this case. This leads us to the first open problem that requires attention.

Open Problem 1. *Consider equation*

$$x_{n+1} = F(x_n), \quad n = 0, 1, \ldots,$$

where $x_0 > 0$, F satisfies hypotheses $(H_1) - (H_4)$, *and assume that condition* $h(b) < a$ *holds.*

(i) *Find an invariant interval, if such exists, which is contained in the interval* $[a, \infty)$.

(ii) *Obtain the detailed description of the basin of attraction of the equilibrium* 0.

In the case when (7) and (11) are satisfied, according to Theorem 5 (iii) all solutions with initial conditions in $[a, \infty)$ are oscillatory with semicycles (positive and negative) of length 1. Also, in Theorem 12, under the same conditions, the existence of attractive period-2 cycles was established. Furthermore, Theorem 10 provides the upper bounds for the length of negative and positive semicycles when conditions (12) and (15) are satisfied, respectively. It is natural to explore further the connections between length of semicycles and the existence of periodic attractive solutions of higher periods. In the example, computer experiments indicated that there is a relationship between the length of semicycles and attractive periodic orbits. So we formulate the following open problem.

Open Problem 2. *Consider equation*

$$x_{n+1} = F(x_n), \quad n = 0, 1, \ldots,$$

where $x_0 > 0$, F satisfies hypotheses $(H_1) - (H_4)$, *and assume that condition* $h(b) \geq a$ *holds.*

(i) *Obtain the conditions for existence and attractivity of periodic solutions of periods greater than* 2.

(ii) *Analyze the existence and the structure of periodic orbits of equation* (5) *with* (6).

Finally, for the example introduced in Section 4, there are several interesting questions to be addressed.

Open Problem 3. *Consider equation*

$$x_{n+1} = \begin{cases} x_n^2/2, & \text{if } x_n \in [0,1) \\ 2\sqrt{2x_n}, & \text{if } x_n \in [1,2) \\ \alpha x_n/(1+x_n), & \text{if } x_n \in [2,\infty) \end{cases}$$

where $x_0 > 0$ and $\alpha > 0$.

(i) *Find the exact value for* α_{cr} *and show that for* $\alpha < \alpha_{cr}$ *all positive solutions are attracted to* 0 *and for* $\alpha > \alpha_{cr}$ *all positive solutions with initial conditions in* $[1,\infty)$ *are trapped in an invariant interval* $I = [2\alpha/3, 4]$.

(ii) *Study bifurcations of the given equation.*

(iii) *Examine the existence, characteristics, and attractivity of periodic cycles.*

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